NON-SINGULAR MULTILINEAR FORMS AND CERTAIN p-WAY MATRIX FACTORIZATIONS*

BY

RUFUS OLDENBURGER

1. **Introduction.** Let there be given a p-way matrix $A=(a_{i_1\cdots i_p})$, $i_1, \cdots, i_p=1, \cdots, n$. The operation which takes A into $A'=(a_{i_1\cdots i_p}b_{i_1j_1})$, where $B=(b_{i_1j_1})$ is a non-singular 2-way matrix and the repeated index indicates summation, is called a *non-singular linear transformation on the index* i_1 *of* A with the matrix B. It is also said to be a non-singular linear transformation on A. If a matrix A' is obtained from A by making non-singular linear transformations on the indices of A with matrices having elements in a field ϕ, then A' is said to be *equivalent* in the field ϕ to A. If A and A' are 2-way matrices this is equivalence in the ordinary sense.

The matrix A is said to be *non-singular* if A is equivalent in some field ϕ, where ϕ contains the elements of A, to $\delta=(\delta_{j_1\cdots j_p})$, where $\delta_{j_1\cdots j_p}=1$ for $j_1=\cdots=j_p=1, \cdots, n$, and $\delta_{j_1\cdots j_p}=0$ if $(j_1, \cdots, j_p)\neq(1, \cdots, 1), \cdots, (n, \cdots, n)$. Similarly a p-way multilinear form

$$G = a_{i_1\cdots i_p}y_{i_1}^{(1)} \cdots y_{i_p}^{(p)} \qquad (i_1, \cdots, i_p = 1, \cdots, n)$$

is said to be non-singular if G is equivalent under non-singular linear transformations

$$y_{i_1}^{(1)} = b_{i_1j_1}^{(1)}x_{j_1}^{(1)}, \cdots, y_{i_p}^{(p)} = b_{i_pj_p}^{(p)}x_{j_p}^{(p)} \qquad (j_1, \cdots, j_p = 1, \cdots, n)$$

to

$$F = \sum_{j_1=\cdots=j_p=1}^{n} x_{j_1}^{(1)} \cdots x_{j_p}^{(p)}.$$

In chapter I of this paper sets of necessary and sufficient conditions, which may be applied in a finite number of steps to a given matrix, are derived for a p-way matrix A, and therefore for its associated form G, to be non-singular. It is necessary in the treatment to distinguish between the cases† where $p=3$ and $p\geqq 4$. Among necessary and sufficient conditions for non-singularity it is proved that a matrix A as given above is non-singular if and only if A can be

* Presented to the Society, March 30 and April 6 and 7, 1934. Abstracts appeared in the Bulletin of the American Mathematical Society, vol. 40 (1934), pp. 219, 226, 227, under the numbers 140, 164, 165 respectively. Received by the editors March 22, 1935.

† The treatment of the case $p=2$ is assumed known.

"factored" into the form $(c^{(1)}_{\alpha i_1} \cdots c^{(p)}_{\alpha i_p})$, where $(c^{(1)}_{\alpha i_1}), \cdots, (c^{(p)}_{\alpha i_p})$ are non-singular 2-way matrices.

The factorization property of a non-singular matrix A suggests the more general problem of determining the conditions under which a matrix can be written in the form $(c^{(1)}_{\alpha i_1} \cdots c^{(p)}_{\alpha i_p})$, $\alpha, i_1, \cdots, i_p = 1, \cdots, n$, where $(c^{(1)}_{\alpha i_1})$ is singular, and $(c^{(2)}_{\alpha i_2}), \cdots, (c^{(p)}_{\alpha i_p})$ are non-singular.*

In the factorizations mentioned above the index α is summed. Necessary and sufficient conditions are also obtained (chapter III) for a matrix $B = (b_{\alpha i_1 \cdots i_p})$ to be of the form $(a^{(1)}_{\alpha\beta i_1} \cdots a^{(p)}_{\alpha\beta i_p})$, α not summed, where the 3-way matrices $(a^{(1)}_{\alpha\beta i_1}), \cdots, (a^{(p)}_{\alpha\beta i_p})$ are non-singular if arrayed as 2-way matrices with $\alpha\beta$ as the row index. The method of treatment applies to the case where $B = (b_{i_1 \cdots i_p})$ factors into the form $(c^{(1)}_{\alpha i_1} \cdots c^{(p)}_{\alpha i_p})$, where α is not summed, and $(c^{(1)}_{\alpha i_1}), \cdots, (c^{(p)}_{\alpha i_p})$ are non-singular.

The terminology and notations used in the ordinary theory of 2-way matrices are assumed known to the reader.

The paper is divided as follows: §1, Introduction. Chapter I, Non-singular multilinear forms: §2, Definitions; §3, Similar transformations; §4, Preliminary theorems; §5, Necessary and sufficient conditions for a matrix to be non-singular; §6, Note on invariant factors. Chapter II, Factorization of p-way matrices into a product of 2-way matrices one of which is singular: §7, Introduction; §8, Canonical diagonal 2-way matrices; §9, Necessary and sufficient conditions for the equivalence of a set of 2-way matrices to a set of diagonal matrices; §10, Necessary and sufficient conditions for the equivalence of a set of p-way matrices, $p \geqq 3$, to a set of diagonal matrices. Chapter III, Factorization of p-way matrices into 3-way matrices: §11, Introduction; §12, Factorization into multiple composites.

Chapter I. Non-singular multilinear forms

2. **Definitions.** The number of elements in the range of an index i is said to be the *order* of i. Thus if i varies over $1, 2, \cdots, n$, then i is of order n. A matrix is said to be of order n if each index is of order n. An ordered set

* In the paper entitled *A new method in the theory of quantics*, Journal of Mathematics and Physics, vol. 8 (1929), pp. 83–84, Hitchcock shows how a matrix of order n associated with a polyadic can always be factored into a sum of products of 2-way matrices, i.e.,

$$(a_{i_1 \cdots i_p}) = \left(\sum_{j=1}^{h} a_{1j i_1} \cdots a_{pj i_p} \right).$$

In his paper entitled *The expression of a tensor or a polyadic as a sum of products*, Journal of Mathematics and Physics, vol. 6 (1927), pp. 164–189, he considers the problem of finding the values of n, p, h for which a matrix can be factored as above into a sum of products of 2-way matrices. He solves a few special cases, but does not solve the general problem.

of indices of a matrix is called a *partition*† of indices. Two partitions T_1, T_2 are said to be equal if they have the same number of indices and corresponding indices are of the same order (the first indices "correspond," the second indices "correspond," etc.). We then write $T_1=T_2$. The product of the orders of the indices in a partition is called the *order of the partition.* An asterisk on T, where T denotes a partition, indicates that the indices of T have been assigned fixed values. For example, if $T=ijk$, and we assign to i, j, k the values 2, 4, 3 respectively, we have $T^*=243$. If $T_1=T_2$ and corresponding indices of T_1 and T_2 have been assigned the same fixed values, we write $T_1^*=T_2^*$.

Let $T_1, T_2, \cdots, T_r$ be mutually exclusive, exhaustive partitions of the indices of a matrix A. The *display* $(a_{T_1 \cdots T_r})$ of A is the matrix obtained by assigning $T_1, \cdots, T_r$ as indices to the different directions of an r-space. The indices in each partition are assumed for convention to vary from right to left; e.g., if $i_1=1, 2$; $i_2=1, 2, 3$; $T=i_1i_2$; then T varies over the range $(i_1i_2)=(11)$, (12), (13), (21), (22), (23).

The $(T_1, \cdots, T_r)$ *diagonal elements* of a matrix $A=(a_{T_1 \cdots T_r})$ in which $T_1=T_2=\cdots=T_r$ are the elements obtained by letting $T_1^*=T_2^*=\cdots=T_r^*$. A matrix $A=(a_{i_1 \cdots i_p})$ is said to be a *diagonal matrix* if its only non-vanishing elements are $(i_1, \cdots, i_p)$ diagonal elements.

A *T-layer* of A is a minor of A obtained by fixing the partition T in the sense that the indices of T are assigned fixed values, and letting the indices of A not contained in T vary over their complete ranges. The *T-rank* of A is the number of linearly independent T-layers of A. A matrix A is said to be *non-singular on* T if the T-layers of A are linearly independent. If a matrix $A=(a_{ijk})$ is non-singular on ij, and k, then A possesses an *inverse* $(A_{ki'j'})=(A_{k'ij})$ *on* ij, k, where

$$(a_{ijk}A_{ki'j'}) = (\delta_{iji'j'}); \quad (a_{ijk}A_{k'ij}) = (\delta_{kk'}).$$

$(\delta_{kk'})$ is a Kronecker delta. $(\delta_{iji'j'})$ displayed in the form $(\delta_{TT'})$, $T=ij$, $T'=i'j'$, is also a Kronecker delta.

A *δ-matrix* on $(T_1, \cdots, T_r)$ is the matrix $(\delta_{T_1 \cdots T_r})$, where $T_1=T_2=\cdots=T_r$ and $\delta_{T_1^* \cdots T_r^*}=1$ when $T_1^*=T_2^*=\cdots=T_r^*$, while $\delta_{T_1^*T_2^* \cdots T_r^*}=0$ otherwise.

The *composite* on T of the matrices $A=(a_{\rho T})$, $B=(b_{T\sigma})$, where ρ, σ, T are partitions, is defined to be the matrix $A|T|B=(a_{\rho T}b_{T\sigma})$. The repetition of the partition T in the last matrix indicates that the indices in T are summed. The matrix $A\times B=(a_{i_1 \cdots i_m}b_{j_1 \cdots j_n})$ is called the *open product* of the matrices $A=(a_{i_1 \cdots i_m})$, $B=(b_{j_1 \cdots j_n})$. A matrix $A=(a_{ijk_1}^{(1)} \cdots a_{ijk_p}^{(p)})$, i not summed (j

† Some of the definitions given in this section are given in *Composition and rank of n-way matrices and multilinear forms*, Annals of Mathematics, vol. 35 (1934), pp. 622–657.

summed), is said to be the *multiple-composite* on the indices i, j of the matrices $(a_{ijk_1}^{(1)}), \cdots, (a_{ijk_p}^{(p)})$.

A matrix $B=(b_{j_1\cdots j_p})$ is said to be composed of a matrix $A=(a_{i_1\cdots i_p})$, $i_s=1, \cdots, n_s$, $s=1, \cdots, p$, *bordered by zeros*, if $b_{j_1\cdots j_p}=a_{i_1\cdots i_p}$ for $j_s=i_s=1, \cdots, n_s$, $s=1, \cdots, p$ while all other elements of B vanish.

Let a set of p-way matrices of order n be given by $B_1, \cdots, B_m$. The *characteristic matrix* of $B_1, \cdots, B_m$ is the matrix $W=(\rho_1 B_1+ \cdots +\rho_m B_m)$ where $\rho_1, \cdots, \rho_m$ are parameters.

3. Similar transformations. Non-singular linear transformations on the sets of variables $x_{j_1}{}^{(1)}, \cdots, x_{j_p}{}^{(1)}$, $j_1, \cdots, j_p=1, \cdots, n$, of

$$F = \sum_{j_1=\cdots=j_p=1}^{n} \overset{(1)}{x}_{j_1} \cdots \overset{(p)}{x}_{j_p},$$

which leave F invariant form a *similar transformation* on F. Two bilinear forms and their associated matrices which are equivalent under a similar transformation are similar in the sense of Dickson.† Let matrices $C_1, \cdots, C_p$ of order n with elements in a field ϕ be given by $(c_{j_1 i_1}^{(1)}), \cdots, (c_{j_p i_p}^{(p)})$ respectively, where these matrices are associated with the transformations

$$(1_1) \qquad \overset{(1)}{x}_{j_1} = \overset{(1)}{c}_{j_1 i_1} \overset{(1)}{y}_{i_1},$$

$$(1_2) \qquad \overset{(2)}{x}_{j_2} = \overset{(2)}{c}_{j_2 i_2} \overset{(2)}{y}_{i_2},$$

$$\cdots\cdots\cdots$$

$$(1_p) \qquad \overset{(p)}{x}_{i_p} = \overset{(p)}{c}_{j_p i_p} \overset{(p)}{y}_{i_p}.$$

Assume that $p \geqq 3$, and that $(1_1), (1_2), \cdots, (1_p)$ leave F invariant, whence

$$(2) \qquad (\overset{(1)}{c}_{\alpha i_1} \cdots \overset{(p)}{c}_{\alpha i_p}) = \delta,$$

where δ is defined on page 422. The layer δ_1 of δ determined by setting $i_1=1$ can be written as $\delta_1=\Gamma C_p$, where

$$\Gamma = (\gamma_{\tau\alpha}) = \begin{pmatrix} \overset{(1)}{c}_{11}\overset{(2)}{c}_{11} \cdots \overset{(p-2)}{c}_{11}\overset{(p-1)}{c}_{11} & \overset{(1)}{c}_{21}\overset{(2)}{c}_{21} \cdots \overset{(p-2)}{c}_{21}\overset{(p-1)}{c}_{21} & \cdots & \overset{(1)}{c}_{n1}\overset{(2)}{c}_{n1} \cdots \overset{(p-2)}{c}_{n1}\overset{(p-1)}{c}_{n1} \\ \overset{(1)}{c}_{11}\overset{(2)}{c}_{11} \cdots \overset{(p-2)}{c}_{11}\overset{(p-1)}{c}_{12} & & & \\ \vdots & & & \cdot \\ \overset{(1)}{c}_{11}\overset{(2)}{c}_{11} \cdots \overset{(p-2)}{c}_{11}\overset{(p-1)}{c}_{1n} & & & \cdot \\ \overset{(1)}{c}_{11}\overset{(2)}{c}_{11} \cdots \overset{(p-2)}{c}_{12}\overset{(p-1)}{c}_{11} & & & \\ \overset{(1)}{c}_{11}\overset{(2)}{c}_{11} \cdots \overset{(p-2)}{c}_{12}\overset{(p-1)}{c}_{12} & & & \cdot \\ \vdots & & & \\ \overset{(1)}{c}_{11}\overset{(2)}{c}_{1n} \cdots \overset{(p-2)}{c}_{1n}\overset{(p-1)}{c}_{1n} & \cdot \quad \cdot & \cdot & \overset{(1)}{c}_{n1}\overset{(2)}{c}_{nn} \cdots \overset{(p-2)}{c}_{nn}\overset{(p-1)}{c}_{nn} \end{pmatrix}.$$

† L. E. Dickson, *Modern Algebraic Theories*, p. 104.

T is the partition $i_2i_3 \cdots i_{p-1}$. The second-order minors of the ξ, χ columns of Γ are of the form

$$M_{\xi\chi} = \overset{(1)}{c}_{\xi 1}\overset{(1)}{c}_{\chi 1} K_{\xi\chi},$$

where

$$K_{\xi\chi} = \begin{vmatrix} \overset{(2)}{c}_{\xi\lambda} \cdots \overset{(p-1)}{c}_{\xi\mu} & \overset{(2)}{c}_{\chi\lambda} \cdots \overset{(p-1)}{c}_{\chi\mu} \\ \overset{(2)}{c}_{\xi\rho} \cdots \overset{(p-1)}{c}_{\xi\sigma} & \overset{(2)}{c}_{\chi\rho} \cdots \overset{(p-1)}{c}_{\chi\sigma} \end{vmatrix}.$$

$K_{\xi\chi}$ is a minor of the display $\theta = (\theta_{\alpha T})$ of the matrix $M = (c^{(2)}_{\alpha i_2} \cdots c^{(p-1)}_{\alpha i_{p-1}})$, α not summed. Let a matrix M' be given by $(C^{(2)}_{i_2\alpha_2} \cdots C^{(p-1)}_{i_{p-1}\alpha_{p-1}})$, where $(C^{(2)}_{i_2\alpha_2}), \cdots, (C^{(p-1)}_{i_{p-1}\alpha_{p-1}})$ are the reciprocals of $(c^{(2)}_{\alpha i_2}), \cdots, (c^{(p-1)}_{\alpha i_{p-1}})$ respectively. Let $\psi = (\psi_{T\beta})$ be the 2-way display of M' obtained by letting the partition T be the index of the rows of ψ and $\beta = \alpha_2\alpha_3 \cdots \alpha_{p-1}$ the index of the columns of ψ. Evidently

$$\theta\psi = \begin{pmatrix} 1 & 0 \cdots 0 \cdots 0 \\ 0 & \cdots 1 \cdots 0 \\ \cdot & \cdot \quad\quad \cdot \\ \cdot & \cdot \quad\quad \cdot \\ 0 & \cdots 0 \cdots 1 \end{pmatrix}.$$

Hence the rank of θ is n.

Since C_p is non-singular and the i_p-rank of δ_1 is 1, the display Γ is of rank 1. Hence for ξ, χ given, the minors $M_{\xi\chi}$ of Γ vanish. Since θ is of rank n as displayed, the minors $K_{\xi\chi}$ do not all vanish for given values of ξ, χ, whence the products $c^{(1)}_{\xi 1}c^{(1)}_{\chi 1}$ vanish for all values of ξ, χ. Take $c^{(1)}_{11} \neq 0$. Then $c^{(1)}_{\chi 1} = 0$ for all $\chi \neq 1$. Similarly take $c^{(1)}_{ii} \neq 0$ for $i = 2, \cdots, n$. Then $c^{(1)}_{\alpha i_1} = 0$ for $\alpha \neq i_1$. This determines a matrix C_1 which satisfies (2). All other matrices $(c^{(1)}_{\alpha i_1})$ which satisfy (2) are obtained from C_1 by arbitrary reordering of the rows and columns of C_1.

Since $c^{(1)}_{\alpha i_1} \cdots c^{(p)}_{\alpha i_p} = 1$ when $i_1 = i_2 = \cdots = i_p$, it follows that if C_1 is a diagonal matrix, $c^{(2)}_{\alpha\alpha}, c^{(3)}_{\alpha\alpha}, \cdots, c^{(p)}_{\alpha\alpha} \neq 0$, and

$$\overset{(p)}{c}_{\alpha\alpha} = \frac{1}{c^{(1)}_{\alpha\alpha} \cdots c^{(p-1)}_{\alpha\alpha}}.$$

Further since $c^{(1)}_{\alpha i} \cdots c^{(p)}_{\alpha i_p} = 0$ when $(i_1, \cdots, i_p) \neq (\alpha, \cdots, \alpha)$, taking $i_1 = \cdots = i_{p-1} \neq i_p$ we get $c^{(p)}_{\alpha i_p} = 0$ when $\alpha \neq i_p$. Similarly, $c^{(2)}_{\alpha i_2} = 0$ when $\alpha \neq i_2, \cdots, c^{(p-1)}_{\alpha i_{p-1}} = 0$ when $\alpha \neq i_{p-1}$.

Evidently all of the solutions of (2) can be obtained from the diagonal matrices $C_1, \cdots, C_p$ as determined above by simultaneous interchanges* of the rows and simultaneous interchanges of the columns of $C_1, \cdots, C_p$. We have proved

THEOREM 1. *If $C_1, \cdots, C_p$ are the p matrices, $p \geqq 3$, of order n associated with a similar transformation in a field ϕ, then $C_1, \cdots, C_p$ are diagonal matrices satisfying the condition $C_p = C_1^{-1} \cdots C_{p-1}^{-1}$ or matrices obtained from these diagonal matrices by simultaneous interchanges of the rows and simultaneous interchanges of the columns of these matrices.*

In the case $p=2$, as is well known, the matrices C_1, C_2 associated with a similar transformation satisfy the property $C_2 = C_1'^{-1}$, C_1' being the transpose of C_1, but are not necessarily diagonal. We have here a case where the theory for p-way matrices, $p \geqq 3$, is much simpler than that for 2-way matrices.

By Theorem 1 the canonical† pairs of p-way matrices, where one of the matrices is a δ-matrix, can be written down.

An effect of a similar transformation on a given matrix is stated in

THEOREM 2. *Under a similar transformation in a field ϕ on the indices $i_1, \cdots, i_p$ the $(i_1, \cdots, i_p)$ diagonal elements of a p-way matrix $A = (a_{i_1 \cdots i_p})$, $p \geqq 3$, are at most rearranged.*

Evidently, $a_{\alpha_1 \cdots \alpha_p} c_{\alpha_1 i}^{(1)} \cdots c_{\alpha_p i}^{(p)} = a_{\alpha \cdots \alpha} c_{\alpha i}^{(1)} \cdots c_{\alpha i}^{(1)}$, where α has some value between 1 and n. α in the above relation is assumed not summed.

4. **Preliminary theorems.** We shall prove

THEOREM 3. *If a p-way matrix $A = (a_{i_1 \cdots i_p})$, $p \geqq 3$, with elements in a field ϕ, is equivalent under non-singular linear transformations in ϕ to a δ-matrix on $(i_1, \cdots, i_p)$, the characteristic matrix $M = (\rho_1 a_{1 i_2 \cdots i_p} + \cdots + \rho_n a_{n i_2 \cdots i_p})$ can be chosen non-singular for at least one set of values of $\rho_1, \cdots, \rho_n$ in ϕ.*

Let $\delta = (\delta_{j_1 \cdots j_p})$ represent a δ-matrix on $(j_1 \cdots j_p)$, and let $\delta_1 = (\delta_{1 j_2 \cdots j_p})$, $\delta_2 = (\delta_{2 j_2 \cdots j_p}), \cdots, \delta_n = (\delta_{n j_2 \cdots j_p})$. Let the characteristic matrix $(\rho_1 \delta_1 + \cdots + \rho_n \delta_n)$ of $\delta_1, \cdots, \delta_n$ be given by $W = (w_{j_2 \cdots j_p})$. The i_1-layers $U_1, \cdots, U_n$ of the matrix $L = (c_{j_1 i_1}^{(1)} \delta_{j_1 \cdots j_p})$ obtained after the non-singular linear transformation (1_1) on $x_{j_1}^{(1)}$ of

$$F = \sum_{j_1 = \cdots = j_p = 1}^{n} x_{j_1}^{(1)} \cdots x_{j_p}^{(p)}$$

are related to $\delta_1, \cdots, \delta_n$ by the relations

* By a simultaneous interchange of the i and j rows of $C_1, \cdots, C_p$ is meant the interchange of the i and j rows of $C_1, \cdots$, the interchange of the i and j rows of C_p.

† "Canonical" is used in the same sense as in the ordinary theory of bilinear forms and 2-way matrices. See Dickson, op. cit., p. 89 ff.

$$U_1 = \overset{(1)}{c_{11}}\delta_1 + \cdots + \overset{(1)}{c_{n1}}\delta_n,$$
$$\cdots\cdots\cdots\cdots$$
$$U_n = \overset{(1)}{c_{1n}}\delta_1 + \cdots + \overset{(1)}{c_{nn}}\delta_n.$$

The characteristic matrix W' of $U_1, \cdots, U_n$ is given by $W'=(\sigma_1 U_1+\cdots+\sigma_n U_n)$. W' can be obtained from W by the non-singular linear transformation on the ρ's given by

$$\rho_1 = \overset{(1)}{c_{11}}\sigma_1 + \cdots + \overset{(1)}{c_{1n}}\sigma_n,$$
$$(3) \qquad \cdots\cdots\cdots\cdots$$
$$\rho_n = \overset{(1)}{c_{n1}}\sigma_1 + \cdots + \overset{(1)}{c_{nn}}\sigma_n.$$

If $\rho_1=\cdots=\rho_n=1$, then W is the non-singular δ-matrix on $(j_2, \cdots, j_p)$. By equations (3), $\sigma_1, \cdots, \sigma_n$ can be so chosen that W' is non-singular.

Non-singular linear transformations $(1_2), \cdots, (1_p)$ on $x_{j_2}^{(2)}, \cdots, x_{j_p}^{(p)}$ associated with L correspond to the transition from $W'=(W'_{j_2\cdots j_p})$ to $W''=(W_{j_2\cdots j_p}c_{j_2i_2}^{(2)}\cdots c_{j_pi_p}^{(p)})$. If $\sigma_1, \cdots, \sigma_n$ are so chosen that $W'=(e_{\xi j_2}^{(2)}\cdots e_{\xi j_p}^{(p)})$, where $(e_{\xi j_2}^{(2)}), \cdots, (e_{\xi j_p}^{(p)})$ are non-singular matrices, then

$$W''=(e_{\xi j_2}^{(2)}c_{j_2i_2}^{(2)}\cdots e_{\xi j_p}^{(p)*}c_{j_pi_p}^{(p)}),$$

which is obviously non-singular. This completes the proof of the theorem.

Theorem 3 gives a necessary condition for the non-singularity of a given matrix. If $p=3$ it is very easy to test a given matrix by means of the theorem. For $p>3$ no simple general method has been found for applying it.

Let $T=(T_1, \cdots, T_m)$ denote a set of 2-way matrices of order n with elements in a field ϕ. If the set T is to be equivalent to a set of diagonal matrices under similar transformation* in ϕ it is necessary that T_1 be equivalent under similar transformation in ϕ to a matrix of the form

$$D = \begin{pmatrix} r_1 I_1 & & & \\ & \cdot & & 0 \\ & 0 & \cdot & \\ & & & r_\mu I_\mu \end{pmatrix},$$

where $r_1, \cdots, r_\mu$ are mutually distinct scalars, and $I_1, \cdots, I_\mu$ are Kronecker deltas. The condition for such equivalence is given in the

LEMMA. *A 2-way matrix T_1 is equivalent under a similar transformation in ϕ to a matrix of type D if and only if the invariant factors of $(T_1-\lambda I)$, where I is a Kronecker delta, split up into distinct linear factors in ϕ.*

---

* For a treatment of the equivalence of 2-way matrices under similar transformations see L. E. Dickson, op. cit., p. 89 ff.

Let the set T now be reduced to $D, T_2', \cdots, T_m'$ under similar transformation in ϕ, and let

$$T_s' = \begin{pmatrix} \overset{s}{T}_{11} & \overset{s}{T}_{12} & \cdots & \overset{s}{T}_{1\mu} \\ \overset{s}{T}_{21} & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \overset{s}{T}_{\mu 1} & \cdot & \cdots & \overset{s}{T}_{\mu\mu} \end{pmatrix} \qquad (s = 2, \cdots, m),$$

where $\overset{s}{T}_{ii}$ is of the same order as I_i for $i=1, \cdots, \mu$. The most general matrix X which satisfies the relation $XDX^{-1}=D$ is of the form

$$X = \begin{pmatrix} X_{11} & & 0 \\ & \ddots & \\ 0 & & X_{\mu\mu} \end{pmatrix},$$

where $X_{\sigma\sigma}$ is of the same order as I_σ for $\sigma=1, \cdots, \mu$. Under similar transformation with X, X^{-1} the matrix T_s' goes into

$$T_s'' = \begin{pmatrix} X_{11}\overset{s}{T}_{11}X_{11}^{-1} & X_{11}T_{12}X_{22}^{-1} & \cdots & X_{11}\overset{s}{T}_{1\mu}X_{\mu\mu}^{-1} \\ X_{22}\overset{s}{T}_{21}X_{11}^{-1} & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ X_{\mu\mu}\overset{s}{T}_{\mu 1}X_{11}^{-1} & \cdot & \cdots & X_{\mu\mu}\overset{s}{T}_{\mu\mu}X_{\mu\mu}^{-1} \end{pmatrix}$$

for $s=2, \cdots, m$. If T_s'' are diagonal matrices, $X_{\alpha\alpha}\overset{s}{T}_{\alpha\beta}X_{\beta\beta}^{-1}=0$ for all s and $\alpha\neq\beta$, whence $\overset{s}{T}_{\alpha\beta}=0$ for all s and $\alpha\neq\beta$. Further, for every σ, the matrices $X_{\sigma\sigma}\overset{2}{T}_{\sigma\sigma}X_{\sigma\sigma}^{-1}, \cdots, X_{\sigma\sigma}\overset{m}{T}_{\sigma\sigma}X_{\sigma\sigma}^{-1}$ must be diagonal matrices. We have proved

THEOREM 4. *Let T_1 of the set $T=(T_1, \cdots, T_m)$ of 2-way matrices of order n satisfy the Lemma. The set T is equivalent under similar transformations in ϕ to a set of diagonal matrices if and only if $\overset{s}{T}_{\alpha\beta}=0$ for $\alpha\neq\beta$, $\alpha\beta=1, \cdots, \mu$, $s=2, \cdots, m$, and the set $\Sigma_\sigma=(\overset{2}{T}_{\sigma\sigma}, \cdots, \overset{m}{T}_{\sigma\sigma})$ is equivalent under similar transformation in ϕ to a set of diagonal matrices for every σ for which the matrices in Σ_σ are of order greater than* 1.

The analogue of Theorem 4 for p-way matrices, $p\geqq 3$, is given in

THEOREM 5. *If $p\geqq 3$, and the p-way matrices $T_1, \cdots, T_m$ of order n with elements in a given field ϕ are equivalent under similar transformation in ϕ to a set of diagonal matrices, the matrices $T_1, \cdots, T_m$ are diagonal matrices.*

By Theorem 2 the $(i_1, \cdots, i_p)$ diagonal elements of a matrix $T_\sigma=(t^\sigma_{i_1\cdots i_p})$ are at most rearranged under similar transformation on $i_1, \cdots, i_p$.

Theorem 5 can be used to test the equivalence of a set $T=(T_1, \cdots, T_m)$ of p-way matrices, $p\geqq 3$, of order n, where T_1 is non-singular, to a set of diagonal matrices. Under reduction of T_1 to δ, where δ is a p-way δ-matrix of order n, the matrices $T_2, \cdots, T_m$ go into a set $\Sigma=(T_{21}, \cdots, T_{m1})$. It is evident that the set T is equivalent to a set of diagonal matrices if and only if the set Σ is equivalent to a set of diagonal matrices under similar transformation.

5. **Necessary and sufficient conditions for a matrix to be non-singular.** We have the following factorization property of non-singular matrices.

THEOREM 6. *A p-way matrix $A=(a_{i_1\cdots i_p})$, $p\geqq 3$, is non-singular if and only if A can be factored into the form $(\overset{(1)}{c}_{\alpha i_1}\cdots\overset{(p)}{c}_{\alpha i_p})$, where $(\overset{(1)}{c}_{\alpha i_1}), \cdots, (\overset{(p)}{c}_{\alpha i_p})$ are non-singular.*

The δ-matrix $(\delta_{j_1\cdots j_p})$ on $(j_1, \cdots, j_p)$ can be written as $(\delta_{\alpha j_1}\cdots\delta_{\alpha j_p})$, where $(\delta_{\alpha j_1}), \cdots, (\delta_{\alpha j_p})$ are Kronecker deltas. If A is non-singular, we have

$$(a_{i_1\cdots i_p}) = (\delta_{j_1\cdots j_p}\overset{(1)}{c}_{j_1 i_1}\cdots\overset{(p)}{c}_{j_p i_p}) = (\delta_{\alpha j_1}\overset{(1)}{c}_{j_1 i_1}\cdots\delta_{\alpha j_p}\overset{(p)}{c}_{j_p i_p}) = (\overset{(1)}{c}_{\alpha i_1}\cdots\overset{(p)}{c}_{\alpha i_p}),$$

where $(\overset{(1)}{c}_{\alpha i_1}), \cdots, (\overset{(p)}{c}_{\alpha i_p})$ are non-singular 2-way matrices.

Every matrix which can be written in the form

$$(\overset{(1)}{c}_{\alpha i_1}\cdots\overset{(p)}{c}_{\alpha i_p}) \qquad (\alpha = 1, \cdots, n), \tag{4}$$

where the rank of $(\overset{(1)}{c}_{\alpha i_1})$ is m and the ranks of $(\overset{(2)}{c}_{\alpha i_2}), \cdots, (\overset{(p)}{c}_{\alpha i_p})$ are all equal to n, can (regardless of the ranges of $i_1, \cdots, i_p$) be reduced under elementary transformations* to

$$N = (\overset{(1)}{c}_{\alpha i_1}\cdots\overset{(p)}{c}_{\alpha i_p}) \qquad (i_1 = 1, \cdots, m;\ \alpha, i_2, \cdots, i_p = 1, \cdots, n),$$

or N bordered by zeros. Our theorems, which will be stated for N instead of (4), will therefore hold for more general cases.

Let E denote the matrix $(e_{i_1\cdots i_p})$; $i_1=1, \cdots, m$; $i_2, \cdots, i_p=1, \cdots, n$. We shall now prove

THEOREM 7. *The matrix E with elements in a field ϕ is factorable into a matrix of type N with elements in ϕ if and only if E is non-singular on i_1, and the i_1-layers of E are equivalent in ϕ to a set of diagonal matrices.*

The i_1-layers of E must be linearly independent since the i_1-rank of N is m and this rank is invariant under non-singular linear transformations.

Let $(\overset{(2)}{C}_{i_2\alpha_2}), \cdots, (\overset{(p)}{C}_{i_p\alpha_p})$ be the reciprocals of $(\overset{(2)}{c}_{\alpha i_1}), \cdots, (\overset{(p)}{c}_{\alpha i_p})$ respectively. If $E=N$, then

$$(e_{i_1\cdots i_p}\overset{(2)}{C}_{i_2\alpha_2}\cdots\overset{(p)}{C}_{i_p\alpha_p}) = (\overset{(1)}{c}_{\alpha i_1}\delta_{\alpha\alpha_2}\cdots\delta_{\alpha\alpha_p}), \tag{5}$$

* For a discussion of elementary transformations see Bôcher, *Introduction to Higher Algebra*, p. 55. Elementary transformations leave the factorization property (4) invariant.

where $(\delta_{\alpha\alpha_2}), \cdots, (\delta_{\alpha\alpha_p})$ are Kronecker deltas of order n. Equation (5) is equivalent to the set

$$
(6) \qquad \begin{aligned}
(e_{1i_2\cdots i_p}C^{(2)}_{i_2\alpha_2}\cdots C^{(p)}_{i_p\alpha_p}) &= (c^{(1)}_{\alpha i_1}\delta_{\alpha\alpha_2}\cdots\delta_{\alpha\alpha_p}),\\
&\cdots\cdots\cdots\\
(e_{mi_2\cdots i_p}C^{(2)}_{i_2\alpha_2}\cdots C^{(p)}_{i_p\alpha_p}) &= (c^{(1)}_{\alpha m}\delta_{\alpha\alpha_2}\cdots\delta_{\alpha\alpha_p}).
\end{aligned}
$$

The matrices $(c^{(1)}_{\alpha 1}\delta_{\alpha\alpha_2}\cdots\delta_{\alpha\alpha_p}), \cdots, (c^{(1)}_{\alpha m}\delta_{\alpha\alpha_2}\cdots\delta_{\alpha\alpha_p})$ are diagonal matrices.

We have proved that if E is factorable into a matrix N, then the i_1-layers of E are linearly independent, and equivalent under non-singular linear transformations in ϕ to a set of diagonal matrices; the converse is simply proved.

We have now determined enough conditions to *test the non-singularity* of a given matrix $A=(a_{i_1\cdots i_p})$, $i_1, \cdots, i_p=1, \cdots, n$. The procedure of this test is as follows. Determine whether or not the necessary condition of Theorem 7 concerning the non-singularity of A on i_1 is satisfied. If A is non-singular on i_1, determine (if possible) whether or not the necessary condition of Theorem 3 is satisfied. If so, choose the matrix M mentioned in Theorem 3 so that it is non-singular. Let the ρ's and the i_1-layers of A be ordered so that $\rho_1\neq 0$. Let $A_1, \cdots, A_n$ designate the i_1-layers of A. Determine by Theorems 4 and 5 whether or not the set $M, A_2, \cdots, A_n$ is equivalent to a set of diagonal matrices. If not, A is singular.* If the contrary is the case, and $p=3$ (similarly for $p\geqq 4$), then there exist non-singular matrices X, Y such that

$$X\rho_iA_iY = D_1,\ XA_2Y = D_2,\ \cdots,\ XA_nY = D_n,$$

where $D_1, \cdots, D_n$ are diagonal matrices. Multiplying the last $(n-1)$ equations by $\rho_2, \cdots, \rho_n$ respectively and subtracting the resulting equations from the first, we obtain $X\rho_1A_1Y=D_1-\sum_{i=2}^{n}\rho_iD_i$. The matrix on the right is a diagonal matrix, from which it follows that the set $A_1, \cdots, A_n$ is equivalent to a set of diagonal matrices. Theorem 7 is now satisfied, and A is non-singular. In certain situations the non-singularity of A is at once evident from Theorem 6.

Let $A_1, \cdots, A_n$ be the i_1-layers of $A=(a_{i_1\cdots i_p})$ with elements in a field ϕ and $p\geqq 4$, and let $\rho_1, \cdots, \rho_n$ be chosen so that the characteristic matrix $M=(\sum_{i=1}^{n}\rho_iA_i)$ is non-singular for $\rho_1\neq 0$. Let the transformations which reduce M to a δ-matrix reduce $A_2, \cdots, A_n$ to $A_2', \cdots, A_n'$. Theorems 5 and 7 imply

THEOREM 8. *The matrix $A=(a_{i_1\cdots i_p})$, $p\geqq 4$, is non-singular if and only if A is non-singular on i_1, and the matrices $A_2', \cdots, A_n'$ are diagonal matrices.*

---

* A matrix not non-singular is said to be *singular*.

If $A=(a_{i_1i_2i_3})$ is a 3-way non-singular matrix of order n and $A_1, \cdots, A_n$ are the i_1-layers of A, by Theorem 7 there exist non-singular matrices X, Y such that

$$XA_iY = Q_i \qquad (i = 1, \cdots, n),$$

where Q_i are diagonal matrices. The matrices X, Y can be obtained from Theorem 4 and the theory of bilinear forms. Arrange the matrices Q_i to form the rows of a 2-way matrix P. Since the i_1-rank of A is n, there exists a non-singular minor V of P. The matrix $V^{-1}P$ is a 2-way display of a matrix δ, where δ is a 3-way δ-matrix of order n. The matrices V'^{-1}, X', Y (the primes here denote transpose) are hence matrices which reduce A to δ under transformation on i_1, i_2, i_3 respectively. The matrices of reduction from a p-way non-singular matrix, $p\geqq 4$, to a δ-matrix are obtained similarly.

We define the *factorization rank* of a matrix $A=(a_{i_1\cdots i_p})$ to be the minimum value of ϵ for which the matrix A can be written in the form $(\sum_{\alpha=1}^{\epsilon} c_{\alpha i_1}^{(1)} \cdots c_{\alpha i_p}^{(p)})$, where $(c_{\alpha i_1}^{(1)}), \cdots, (c_{\alpha i_p}^{(p)})$ are 2-way matrices. This rank is invariant under non-singular linear transformations. The factorization rank of a matrix $(\sum_{\alpha=1}^{\epsilon} c_{\alpha i_1}^{(1)} \cdots c_{\alpha i_p}^{(p)})$ is n if all of the matrices $(c_{\alpha i_1}^{(1)}), \cdots, (c_{\alpha i_p}^{(p)})$ are non-singular.

In another paper* the author has defined certain ranks of a p-way matrix which are invariant under non-singular linear transformations. The following theorem is easily proved.

THEOREM 9. *A p-way matrix A of order n is non-singular if and only if all of its invariant ranks are equal to n.*

6. **Note on invariant factors.** The matrix $W=(\sum_{i=1}^{m}\rho_iB_i)$ used in §2 suggests the following generalization of ordinary invariant factor theory. Let B_i, $i=1, \cdots, m$, be square matrices of order n. Let G_t be the greatest common divisor of the minors of W of the tth order, and let $G_0=1$. We define the tth *invariant factor* of W to be the quotient G_t/G_{t-1}. It is determined up to a constant factor. It is assumed in factoring the minors of M to obtain the G_t, $t=0, 1, \cdots, n$, that the factorization is performed in a given field.† Now when $B_1, \cdots, B_m$ are multiplied by non-singular matrices the quotients G_t/G_{t-1} are invariant. If

$$B_1' = a_{11}B_1 + \cdots + a_{1m}B_m,$$
$$\cdots\cdots\cdots\cdots$$
$$B_m' = a_{1m}B_1 + \cdots + a_{mm}B_m,$$

* *Composition and rank of n-way matrices and multilinear forms*, Annals of Mathematics, vol. 35 (1934), pp. 625, 633, 634.

† It is to be noted that for $m\geqq 3$ a minor of W cannot always be factored into distinct linear factors in a field. To obtain linear factors it is necessary in general to use the quasi-field of quaternions and other generalizations of fields.

and the matrix of coefficients is non-singular, then the invariant factors of $W' = (\alpha_1 B_1' + \cdots + \alpha_m B_m')$ can be obtained from the invariant factors of W by the transformations

$$\rho_1 = a_{11}\alpha_1 + \cdots + a_{m1}\alpha_m,$$
$$\cdots\cdots\cdots\cdots$$
$$\rho_m = a_{1m}\alpha_1 + \cdots + a_{mm}\alpha_m.$$

Hence powers of terms occurring in invariant factors go into like powers under the transformation from the set $B_1, \cdots, B_m$ to $B_1', \cdots, B_m'$.

The determinant of W in the proof of Theorem 3 where $p=3$ is the only invariant factor of W distinct from unity. This determinant factors into distinct linear factors in any field ϕ. It follows that if Δ is the determinant of the characteristic matrix of the i-layers of $R=(r_{ijk})$ and R is equivalent in ϕ to a δ-matrix on (i, j, k), then Δ factors into distinct linear factors in ϕ, and is the only invariant factor distinct from a constant.

An exact generalization of the theory of this section holds for p-way matrices $p \geqq 3$, where the invariant factors are defined in terms of space determinants.

Chapter II. Factorization of p-way matrices into a product of 2-way matrices one of which is singular

7. **Introduction.** By Theorem 7, a matrix $A=(a_{i_1\cdots i_p})$; $i_1=1, \cdots, m$; $i_2, \cdots, i_p=1, \cdots, n$, can be factored into a matrix $A=(c^{(1)}_{\alpha i_1} \cdots c^{(p)}_{\alpha i_p})$; $\alpha=1, \cdots, n$, where $(c^{(1)}_{\alpha i_1}), \cdots, (c^{(p)}_{\alpha i_p})$ are non-singular on $i_1, \cdots, i_p$ respectively, if and only if the i_1-layers of $A_1, \cdots, A_m$ of A are linearly independent, and are simultaneously equivalent to a set of diagonal matrices. To complete the treatment of factorizations of the above type where $(c^{(1)}_{\alpha i_1})$ is singular we must obtain necessary and sufficient conditions for the equivalence of $A_1, \cdots, A_m$ to a set of diagonal matrices. We do not assume as in the treatment of a non-singular matrix A that there exist values of $\rho_1, \cdots, \rho_m$ not all zero such that $\rho_1 A_1 + \cdots + \rho_m A_m$ is non-singular.

Using an essentially different technique we derive first (Theorem 10) the canonical diagonal 2-way matrices $C_1, \cdots, C_q$ to which diagonal matrices $E_1, \cdots, E_q$ are equivalent under non-singular linear transformations, and then obtain necessary and sufficient conditions (Theorems 12, 13) for the equivalence of a set of 2-way matrices $C_1, \cdots, C_i, F_{i+1}$ to a set of diagonal matrices.

To test the equivalence of a set $S=(A_1, \cdots, A_m)$ of 2-way matrices to a set of diagonal matrices, reduce A_1 to a canonical diagonal matrix C_1. If this is not possible (this is very easy to determine), it follows that the set S

is not equivalent to a set of diagonal matrices. If A_1 is equivalent to a matrix C_1, reduce A_1 to C_1. The remaining matrices of S are simultaneously transformed into a set $A_2', \cdots, A_m'$ respectively. Apply Theorems 12 and 13 to determine whether or not the pair C_1, A_2' is equivalent to a pair of canonical diagonal matrices C_1, C_2. If not, the set S is not equivalent to a set of diagonal matrices. If on the other hand the pair C_1, A_2' is equivalent to a pair C_1, C_2, reduce C_1, A_2' to C_1, C_2. The remaining matrices in the set S are then transformed into a set $A_3'', \cdots, A_m''$. Now apply Theorems 12 and 13 to C_1, C_2, A_3''. Continue this process until one finally arrives at a set of canonical diagonal matrices $C_1, \cdots, C_m$ to which the set S is equivalent.

8. **Canonical diagonal 2-way matrices.** Adopting the notation used by J. Williamson in a recent paper* we shall write the "diagonal block matrix"

$$C = \begin{Vmatrix} C_1 & & \\ & \cdot & 0 \\ & \cdot & \\ 0 & \cdot & \\ & & C_s \end{Vmatrix},$$

where $C_1, \cdots, C_s$ are square minors of C, in the form

$$C = [C_1, \cdots, C_s].$$

Let the letters I, ρ with superscripts and subscripts denote a Kronecker delta, and a parameter respectively. We shall prove

THEOREM 10. *Let $S=(E_1, \cdots, E_q)$ denote a set of diagonal 2-way matrices of order n with elements in a given field ϕ. The set S is equivalent in ϕ to a set $S'=(C_1, \cdots, C_q)$, where*

$$(7_1) \qquad C_1 = [I_1^1, 0],$$

$$(7_2) \qquad C_2 = [S_1^1, S_2^1],$$

and

$$(8_1) \qquad S_1^1 = [\rho_1^2 I_1^2, \cdots, \rho_t^2 I_t^2],$$

$$(8_2) \qquad S_2^1 = [I_{t+1}^2, 0];$$

the $\rho_1^2, \cdots, \rho_t^2$ are all distinct, and the matrix S_1^1 is of the same order as I_1^1; in general any pair in the set S' is of the form

$$(7_i) \qquad C_i = [\rho_1^i I_1^i, \cdots, \rho_{s(i)-1}^i I_{s(i)-1}^i, 0 \cdot I_{s(i)}^i],$$

$$(7_{i+1}) \qquad C_{i+1} = [S_1^i, \cdots, S_{s(i)}^i],$$

where

* *Simultaneous reduction of two matrices to triangle form*, American Journal of Mathematics, vol. 57 (1935), p. 282.

$$(9_1)\qquad S_1^i = [\overset{i+1}{\rho_1}\overset{i+1}{I_1}, \cdots, \overset{i+1}{\rho_{m_1}}\overset{i+1}{I_{m_1}}],$$

$$(9_2)\qquad S_2^i = [\overset{i+1}{\rho_{m_1+1}}\overset{i+1}{I_{m_1+1}}, \cdots, \overset{i+1}{\rho_{m_1+m_2}}\overset{i+1}{I_{m_1+m_2}}],$$

. . . .

$$(9_{s(i)-1})\quad S_{s(i)-1}^i = [\overset{i+1}{\rho_{m_1+\cdots+m_{s(i)-2}+1}}\overset{i+1}{I_{m_1+\cdots+m_{s(i)-2}+1}}, \cdots, \overset{i+1}{\rho_{m_1+\cdots+m_{s(i)-1}}}\overset{i+1}{I_{m_1+\cdots+m_{s(i)-1}}}],$$

$$(9_{s(i)})\qquad S_{s(i)}^i = [\overset{i+1}{I_{m_1+\cdots+m_{s(i)-1}+1}}, 0],$$

and where the ρ's in each matrix $S_1^i, \cdots, S_{s(i)-1}^i$ are distinct, and the orders of the matrices $S_1^i, \cdots, S_{s(i)}^i$ are equal to the orders of $I_1^i, \cdots, I_{s(i)}^i$ respectively.

At the same time we shall prove

THEOREM 11. *The matrices X, Y which satisfy the relations*

$$XC_1Y = C_1, \cdots, XC_iY = C_i$$

are of the form

$$(10)\qquad X = \begin{bmatrix} X_{11} & & & X_{1,s(i)} \\ & \ddots & 0 & \vdots \\ 0 & & X_{s(i)-1,s(i)-1} & \\ & & & X_{s(i),s(i)} \end{bmatrix},$$

$$Y = \begin{bmatrix} X_{11}^{-1} & & & \\ & \ddots & & 0 \\ 0 & & X_{s(i)-1,s(i)-1}^{-1} & \\ Y_{s(i),1} & \cdots & & Y_{s(i),s(i)} \end{bmatrix},$$

where $X_{11}, \cdots, X_{s(i),s(i)}$ are of the same orders as $I_1^i, \cdots, I_{s(i)}^i$ respectively.

If E_1 is of rank r, E_1 is obviously equivalent to C_1, where I_1^1 is of rank r. It is readily verified that if $XC_1Y = C_1$, then

$$X = \begin{pmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{pmatrix}, \qquad Y = \begin{pmatrix} X_{11}^{-1} & 0 \\ Y_{21} & Y_{22} \end{pmatrix},$$

where X_{11} is a minor of order r.

Assume now that $E_1, \cdots, E_i$ have been reduced to canonical forms $C_1, \cdots, C_i$ and that X, Y are as given in (10). Let E_{i+1} be denoted by

$$[\overset{i+1}{P_i}, \cdots, \overset{i+1}{P_{s(i)}}],$$

where the minors $P_1^{i+1}, \cdots, P_{s(i)}^{i+1}$ are of the same orders as $I_1^i, \cdots, I_{s(i)}^i$. Let $X_{1,s(i)}, \cdots, X_{s(i)-1,s(i)}, Y_{s(i),1}, \cdots, Y_{s(i),s(i)-1}$ be set equal to zero. Then

$$XE_{i+1}Y = [X_{11}P_1^{i+1}X_{11}^{-1}, \cdots, X_{s(i)-1,s(i)-1}P_{s(i)-1}^{i+1}X_{s(i)-1,s(i)-1}^{-1}, X_{s(i),s(i)}P_{s(i)}^{i+1}Y_{s(i),s(i)}].$$

Choose the non-zero minors of X, Y so that

$$X_{11}P_1^{i+1}X_{11}^{-1}, \cdots, X_{s(i)-1,s(i)-1}P_{s(i)-1}^{i+1}X_{s(i)-1,s(i)-1}^{-1}$$

are in the classical canonical forms* $S_1^i, \cdots, S_{s(i)-1}^i$, and choose $X_{s(i),s(i)}$, $Y_{s(i),s(i)}$ so that $X_{s(i),s(i)}P_{s(i)}^{i+1}Y_{s(i),s(i)}$ is a Kronecker delta bordered by zeros. The matrix E_{i+1} has now been brought into a canonical form of type C_{i+1}.

It is readily verified that the matrices X, Y of (10) which satisfy the relation

$$XC_{i+1}Y = C_{i+1}$$

are of the form

$$X' = \begin{pmatrix} X'_{11} & & & & & X'_{1,m_1+\cdots+m_{s(i)}} \\ & \ddots & & 0 & & \cdot \\ & & X'_{m_1,m_1} & & & \cdot \\ & 0 & & X'_{m_1+1,m_1+1} & & \cdot \\ & & & & \ddots & \\ & & & & & X'_{m_1+\cdots+m_{s(i)},m_1+\cdots+m_{s(i)}} \end{pmatrix},$$

$$Y' = \begin{pmatrix} X'^{-1}_{11} & & & & & \\ & \ddots & & & 0 & \\ & & X'^{-1}_{m_1,m_1} & & & \\ & & & X'^{-1}_{m_1+1,m_1+1} & & \\ & 0 & & & \ddots & \\ & & & & X'^{-1}_{m_1+\cdots+m_{s(i)}-1,m_1+\cdots+m_{s(i)}-1} & \\ Y'_{m_1+\cdots+m_{s(i)},1} & & & & \cdots & Y'_{m_1+\cdots+m_{s(i)},m_1+\cdots+m_{s(i)}} \end{pmatrix}$$

where $m_{s(i)}=2$, and $X'_{11}, \cdots, X'_{m_1+\cdots+m_{s(i)}-1,m_1+\cdots+m_{s(i)}-1}$ are of the same orders as $I_1^{i+1}, \cdots, I_{m_1+\cdots+m_{s(i)}-1}^{i+1}$ respectively. Theorems 10 and 11 now follow by induction.

9. **Necessary and sufficient conditions for the equivalence of a set of 2-way matrices to a set of diagonal matrices.** Two-way matrices $F_1, \cdots, F_m$ are equivalent under non-singular linear transformations to a set of diagonal

---

* This reduction is accomplished by rearranging the diagonal elements of E_{i+1}. Hence the reduction of E_{i+1} can be accomplished in any field ϕ.

matrices if and only if $F_1, \cdots, F_m$ are equivalent to a set of canonical diagonal matrices $C_1, \cdots, C_m$ as given in Theorem 10. We shall therefore prove Theorems 12 and 13 below.

Write

$$F_\mu = \begin{pmatrix} A^\mu_{11} & A^\mu_{12} \cdots A^\mu_{1,s(i)} \\ A^\mu_{21} & \cdot \quad \cdot \cdot \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \cdot \cdot \quad \cdot \\ A^\mu_{s(i),1} & \cdot \quad \cdot \cdot \cdot \; A^\mu_{s(i),s(i)} \end{pmatrix} \qquad (\mu = i+1, \cdots, m),$$

where $A^\mu_{11}, A^\mu_{22}, \cdots, A^\mu_{s(i),s(i)}$ are of the same orders as $I_1^i, I_2^i, \cdots, I^i_{s(i)}$ respectively. These last matrices are minors of the matrix C_i of Theorem 10.

THEOREM 12. *Let $\Sigma = (C_1, \cdots, C_i, F_{i+1}, \cdots, F_m)$ be a set of 2-way matrices of order n with elements in a field ϕ. If $A^\mu_{s(i),s(i)} = 0$ for $\mu = i+1, \cdots, m$, the set Σ is equivalent in ϕ to a set of diagonal matrices if and only if the following conditions are satisfied:*

(a) $A^\mu_{\alpha\beta} = 0$; $\alpha \neq \beta$; $\alpha, \beta = 1, \cdots, s(i)$, *and* $\mu = i+1, \cdots, m$.

(b) $A^{i+1}_{\gamma\gamma}, \cdots, A^m_{\gamma\gamma}$ *are equivalent in ϕ for every γ in the set $\gamma = 1, \cdots, s(i)-1$ to a set of diagonal matrices under similar transformation.*

Let X, Y be given as in (10). We shall denote the matrices $F_\mu' = XF_\mu Y$ by

$$F_\mu' = \begin{pmatrix} B^\mu_{11} & B^\mu_{12} \cdots B^\mu_{1,s(i)} \\ B^\mu_{21} & \cdot \quad \cdot \cdot \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \cdot \cdot \quad \cdot \\ B^\mu_{s(i),1} & \cdot \quad \cdot \cdot \cdot \; B^\mu_{s(i),s(i)} \end{pmatrix} \qquad (\mu = i+1, \cdots, m),$$

where the minors B^μ_{kl} are of the same orders as A^μ_{kl} for $k, l = 1, \cdots, s(i)$. Now $B^\mu_{\alpha,s(i)} = X_{\alpha\alpha} A^\mu_{\alpha,s(i)} Y_{s(i),s(i)}, \alpha = 1, \cdots, s(i)-1$. Also $B^\mu_{s(i),\alpha} = X_{s(i),s(i)} A^\mu_{s(i),\alpha} X^{-1}_{\alpha\alpha}$, $\alpha = 1, \cdots, s(i)-1$. If the matrices F_μ' are diagonal matrices it is necessary that $B^\mu_{\alpha,s(i)} = B^\mu_{s(i),\alpha} = 0$ for $\alpha = 1, \cdots, s(i)-1$, whence $A^\mu_{\alpha,s(i)} = A^\mu_{s(i),\alpha} = 0$, $\alpha = 1, \cdots, s(i)-1$. Substituting these results in the formulas for the remaining elements of F_μ we find we must also have $A^\mu_{\alpha\beta} = 0$ for $\alpha \neq \beta$; $\alpha, \beta = 1, \cdots, s(i)-1$, and the set of matrices $A^{i+1}_{\gamma\gamma}, \cdots, A^m_{\gamma\gamma}$ must be equivalent for every γ, where $\gamma = 1, \cdots, s(i)-1$, to a set of diagonal matrices under similar transformation. Necessary and sufficient conditions for such equivalence are given in Theorem 4. We have proved the necessity of the conditions of Theorem 12. They are also evidently sufficient.

If $A^{i+1}_{s(i),s(i)}$ is of rank $r' \neq 0$, the matrix F_{i+1} can be reduced under transformations leaving $C_1, \cdots, C_i$ invariant to a new matrix F_{i+1} in which

$$A^{i+1}_{s(i),s(i)} = [I^{i+1}_{s(i+1)-1}, 0], \tag{11}$$

where $I^{i+1}_{s(i+1)-1}$ is a Kronecker delta of order r'. Write

$$A^{i+1}_{\rho,s(i)} = (A^{[1]}_{\rho,s(i)} A^{[2]}_{\rho,s(i)}), \qquad A^{i+1}_{s(i),\rho} = \begin{pmatrix} A^{[1]}_{s(i),\rho} \\ A^{[2]}_{s(i),\rho} \end{pmatrix},$$

where $\rho = 1, \cdots, s(i)-1$, and the $A^{[1]}_{\rho,s(i)}$ have r' columns while the $A^{[1]}_{s(i),\rho}$ have r' rows. We can now state

THEOREM 13. *Let the minor $A^{i+1}_{s(i),s(i)}$ of F_{i+1} be reduced as in* (11). *The set of matrices $\Sigma' = (C_1, \cdots, C_i, F_{i+1})$ is equivalent in a field ϕ to a set of diagonal matrices if and only if the following conditions are satisfied*:

(a) $A^{[2]}_{s(i),\rho} = A^{[2]}_{\rho,s(i)} = 0$ *for* $\rho = 1, \cdots, s(i)-1$.

(b) $A^{i+1}_{\alpha\beta} = A^{[1]}_{\alpha,s(i)} A^{[1]}_{s(i),\beta}$; $\alpha \neq \beta$; $\alpha, \beta = 1, \cdots, s(i)-1$.

(c) *The invariant factors of the matrices* $A^{i+1}_{\alpha\alpha} - A^{[1]}_{\alpha,s(i)} A^{[1]}_{s(i),\alpha} - \lambda I_\alpha$, $\alpha = 1, \cdots, s(i)-1$, *where the I_α are Kronecker deltas, factor into distinct linear factors in ϕ.*

The non-singular matrices $X_{s(i),s(i)}$, $Y_{s(i),s(i)}$ which transform $A^{i+1}_{s(i),s(i)} = [I^{i+1}_{s(i+1)-1}, 0]$ into itself, so that $X_{s(i),s(i)} A^{i+1}_{s(i),s(i)} Y_{s(i),s(i)} = A^{i+1}_{s(i),s(i)}$, are of the form

$$X_{s(i),s(i)} = \begin{pmatrix} W_{11} & W_{12} \\ 0 & W_{22} \end{pmatrix}, \qquad Y_{s(i),s(i)} = \begin{pmatrix} W_{11}^{-1} & 0 \\ V_{21} & V_{22} \end{pmatrix},$$

where W_{11} is a minor of the same order as $I^{i+1}_{s(i+1)-1}$. If the set Σ' is to be equivalent to a set of diagonal matrices, it must be equivalent to a set $C_1, \cdots, C_{i+1}$, where $S^i_{s(i)} = A^{i+1}_{s(i),s(i)}$. The matrix $S^i_{s(i)}$ is a minor of C_{i+1} as in Theorem 10. The matrix F_{i+1} must then be equivalent to a matrix C_{i+1} under transformations X, Y which leave the set $\Sigma'' = (C_1, \cdots, C_i, A^{i+1}_{s(i),s(i)})$ invariant. If such matrices X, Y exist, then X^{-1}, Y^{-1} also leaves Σ'' invariant. If $XF_{i+1}Y = C_{i+1}$, then $F_{i+1} = X^{-1}C_{i+1}Y^{-1}$. Hence the set Σ' is equivalent to a set $C_1, \cdots, C_{i+1}$ if and only if there exist matrices X, Y leaving Σ'' invariant such that

$$F_{i+1} = XC_{i+1}Y. \tag{12}$$

If there are to be matrices X, Y, C_{i+1} such that (12) is satisfied, it is readily seen by equating matrices that the following conditions must be satisfied:

$$A^{i+1}_{s(i),1} = \begin{pmatrix} W_{11} & 0 \\ 0 & 0 \end{pmatrix} Y_{s(i),1}; \cdots ; A^{i+1}_{s(i),s(i)-1} = \begin{pmatrix} W_{11} & 0 \\ 0 & 0 \end{pmatrix} Y_{s(i),s(i)-1}; \tag{13}$$

$$A^{i+1}_{1,s(i)} = X_{1,s(i)} \begin{pmatrix} W_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix}; \cdots ; A^{i+1}_{s(i)-1,s(i)} = X_{s(i)-1,s(i)} \begin{pmatrix} W_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \tag{14}$$

Let

$$Y_{s(i),\rho} = \begin{pmatrix} Y_\rho' \\ Y_\rho'' \end{pmatrix}, \qquad X_{\rho,s(i)} = (X_\rho' \, X_\rho'')$$

for $\rho = 1, \cdots, s(i)-1$, where Y_ρ' has r' rows and X_ρ' has r' columns. By (13) and (14) we have

$$\overset{[2]}{A}_{s(i),\rho} = \overset{[2]}{A}_{\rho,s(i)} = 0 \qquad (\rho = 1, \cdots, s(i)-1). \tag{15}$$

Further

$$\begin{aligned} Y_1' &= W_{11}^{-1}\overset{[1]}{A}_{s(i),1}, \cdots, Y_{s(i)-1}' = W_{11}^{-1}\overset{[1]}{A}_{s(i),s(i)-1}; \\ X_1' &= \overset{[1]}{A}_{1,s(i)}W_{11}, \cdots, X_{s(i)-1}' = \overset{[1]}{A}_{s(i)-1,s(i)}W_{11}. \end{aligned} \tag{16}$$

Substituting (16) in $XC_{i+1}Y$, and using (15), we get

$$XC_{i+1}Y = \begin{pmatrix} (X_{11}S_1^iX_{11}^{-1} + \overset{[1]}{A}_{1,s(i)}\overset{[1]}{A}_{s(i),1}) & \overset{[1]}{A}_{1,s(i)}\overset{[1]}{A}_{s(i),2} & \cdots & \overset{[1]}{A}_{1,s(i)}\overset{[1]}{A}_{s(i),s(i)-1} & \cdots & \overset{i+1}{A}_{1,s(i)} \\ \overset{[1]}{A}_{2,s(i)}\overset{[1]}{A}_{s(i),1} & \cdot & \cdots & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdots & \cdot \\ \overset{[1]}{A}_{s(i)-1,s(i)}\overset{[1]}{A}_{s(i),1} & \cdot & \cdots & (X_{s(i)-1,s(i)-1}S_{s(i)-1}^iX_{s(i)-1,s(i)-1}^{-1} + \overset{[1]}{A}_{s(i)-1,s(i)}\overset{[1]}{A}_{s(i),s(i)-1}) & \cdots & \overset{i+1}{A}_{s(i)-1,s(i)} \\ \overset{i+1}{A}_{s(i),1} & \cdot & \cdots & \overset{i+1}{A}_{s(i),s(i)-1} & \cdots & \overset{i+1}{A}_{s(i),s(i)} \end{pmatrix}.$$

To have the above matrix equal to F_{i+1} we must have

$$\overset{i+1}{A}_{\alpha\beta} = \overset{[1]}{A}_{\alpha,s(i)}\overset{[1]}{A}_{s(i),\beta}; \qquad \alpha \neq \beta; \qquad \alpha, \beta = 1, \cdots, s(i)-1.$$

Also $X_{\alpha\alpha}S_\alpha^iX_{\alpha\alpha}^{-1} + \overset{[1]}{A}_{\alpha,s(i)}\overset{[1]}{A}_{s(i),\alpha} = \overset{i+1}{A}_{\alpha\alpha}$, whence $\overset{i+1}{A}_{\alpha\alpha} - \overset{[1]}{A}_{\alpha,s(i)}\overset{[1]}{A}_{s(i),\alpha}$ must each be equivalent under similar transformation to a diagonal matrix for every α where $\alpha = 1, \cdots, s(i)-1$. The necessity of condition (c) of the theorem now follows from the lemma, §4.

We have proved the necessity of the conditions of Theorem 13. The sufficiency of these conditions is evident.

10. Necessary and sufficient conditions for the equivalence of a set of p-way matrices, $p \geqq 3$, to a set of diagonal matrices. We shall now state the analogues of Theorems 10 and 11 for $p \geqq 3$.

Let $C_1, \cdots, C_s$ be p-way matrices, $p \geqq 3$, of orders $n_1, \cdots, n_s$ respectively. Let a p-way matrix C of order $n_1 + \cdots + n_s$ be constructed in a p-space by placing the matrices $C_1, \cdots, C_s$ in a non-overlapping fashion on the principal diagonal of C so that the principal diagonals of $C_1, \cdots, C_s$ form the principal diagonal of C, and let the elements of C not in these "minors" be zero. The matrix C will be denoted, as in the 2-way case, by $C = [C_1, \cdots, C_s]$. Let the quantities λ, δ with or without subscripts and superscripts denote a p-way non-singular diagonal matrix and a p-way δ-matrix respectively. We can now state

THEOREM 14. *Let $S = [E_1, \cdots, E_q]$ be a set of p-way diagonal matrices, $p \geqq 3$, of order n with elements in a field ϕ. The set S is equivalent in ϕ to a set $S' = (C_1, \cdots, C_q)$, where*

$$C_1 = [\delta_1, 0], \qquad C_2 = [S_1, T_1],$$

and

$$S_1 = [\lambda_1^2, 0], \qquad T_1 = [\delta_2, 0];$$

the matrix S_1 is of the same order as δ_1; in general any consecutive pair in the set S' is of the form

$$C_\alpha = [\lambda_1^\alpha, 0, \lambda_2^\alpha, 0, \cdots, \lambda_{\sigma(\alpha)}^\alpha, 0, \delta_\alpha, 0],$$

$$C_{\alpha+1} = [Q_1^\alpha, R_1^\alpha, Q_2^\alpha, R_2^\alpha, \cdots, Q_{\sigma(\alpha)}^\alpha, R_{\sigma(\alpha)}^\alpha, S_\alpha, T_\alpha],$$

where

$$Q_q^\alpha = [\lambda_{2q-1}^{\alpha+1}, 0] \qquad (q = 1, \cdots, \sigma(\alpha)),$$

$$R_r^\alpha = [\lambda_{2r}^{\alpha+1}, 0] \qquad (r = 1, \cdots, \sigma(\alpha)),$$

$$S_\alpha = [\lambda_{\sigma(\alpha+1)}^{\alpha+1}, 0],$$

$$T_\alpha = [\delta_{\alpha+1}, 0],$$

and $\sigma(\alpha) = 2^{\alpha-1} - 1$, $\alpha \geqq 2$, and the minors $Q_1^\alpha, R_1^\alpha, \cdots, Q_{\sigma(\alpha)}^\alpha, R_{\sigma(\alpha)}^\alpha, S_\alpha, T_\alpha$ are of the same orders as the corresponding minors $\lambda_1^\alpha, 0, \cdots, \lambda_{\sigma(\alpha)}^\alpha, 0, \delta_\alpha, 0$ of C_α.

Let $i, j, \cdots, k, l$ be the indices of the matrices $C_1, \cdots, C_q$. As we prove Theorem 14 we shall also prove

THEOREM 15. *Let $A^q = (a_{i\beta}^q)$, $B^q = (b_{j\gamma}^q)$, $\cdots$, $C^q = (c_{k\tau}^q)$, $D^q = (d_{l\epsilon}^q)$ denote matrices which, under non-singular linear transformations with these matrices on the indices $i, j, \cdots, k, l$ of the matrices in the set $S' = (C_1, \cdots, C_q)$, leave the set S' invariant or at most reorder the elements of the minors λ_m^q, $m = 1, \cdots, \sigma(q)$, of C_q independently for each m. The matrices $A^q, \cdots, D^q$ are of the form*

$$A^q = \begin{pmatrix} A^q_{11} & & & & & & 0 \\ & A^q_{12} & & & & & \\ & & A^q_{21} & & & & \\ & & & A^q_{22} & & & \\ & 0 & & & \ddots & & \\ & & & & & A^q_{\sigma(q)+1,1} & \\ & & A^q_{\sigma(q)+1,3} & & & & A^q_{\sigma(q)+1,2} \end{pmatrix}, \cdots,$$

(17)

$$D^q = \begin{pmatrix} D^q_{11} & & & & & & 0 \\ & D^q_{12} & & & & & \\ & & D^q_{21} & & & & \\ & & & D^q_{22} & & & \\ & 0 & & & \ddots & & \\ & & & & & D^q_{\sigma(q)+1,1} & \\ & & D^q_{\sigma(q)+1,3} & & & & D^q_{\sigma(q)+1,2} \end{pmatrix},$$

where $D^q_{\gamma 1} = A^{q-1}_{\gamma 1} B^{q-1}_{\gamma 1} \cdots C^{q-1}_{\gamma 1}$ *for* $\gamma = 1, \cdots, \sigma(q)+1$; $D^q_{\gamma 2} = A^{q-1}_{\gamma 2} B^{q-1}_{\gamma 2} \cdots C^{q-1}_{\gamma 2}$ *for* $\gamma = 1, \cdots, \sigma(q)$; *and corresponding minors in the sets* $(A^q_{11}, \cdots, A^q_{\sigma(q)+1,1})$, $\cdots$, $(D^q_{11}, \cdots, D^q_{\sigma(q)+1,1})$ *are of the same orders as* $\lambda_1{}^q, \lambda_2{}^q, \cdots, \lambda^q_{\sigma(q)}, \delta_q$ *respectively; the remaining minors of* $A^q, \cdots, D^q$ *are of the same orders as the corresponding zero minors of* C_q; *also, every minor except the last on the diagonal of each matrix* $A^q, \cdots, D^q$ *as written in* (17) *is a diagonal matrix. The matrices* $A^q, \cdots, D^q$ *may also be of the types obtained from those given in* (17) *by simultaneous interchanges of the rows and simultaneous interchanges of the columns of the minors in each of the sets* $(A^q_{\sigma(q)+1,1}, \cdots, D^q_{\sigma(q)+1,1})$, $(A^q_{\alpha\beta}, \cdots, D^q_{\alpha\beta})$; $\alpha = 1, \cdots, \sigma(q)$; $\beta = 1, 2$, *these interchanges being made independently for each set.*

If there are u^1_1 non-vanishing elements on the diagonal of E_1, it is evident at once that E_1 is equivalent to C_1, where δ_1 is of order u^1_1.

Denote C_α by $(c^\alpha_{ij\cdots kl})$ for every α. If $(c^1_{ij\cdots kl} a^1_{i\beta} b^1_{j\gamma} \cdots c^1_{k\tau} d^1_{l\epsilon}) = (c^1_{ij\cdots kl})$, then

$$\left(\sum_{\alpha=1}^{u^1_1} a^1_{\alpha\beta} b^1_{\alpha\gamma} \cdots c^1_{\alpha\tau} d^1_{\alpha\epsilon} \right) = 0 \tag{18}$$

for $\gamma, \cdots, \tau, \epsilon = 1, \cdots, n$ and $\beta = u^1_1 + 1, \cdots, n$. By (18), $QP = 0$ where

$$Q = \begin{Vmatrix} (\overset{1}{b}_{11}\cdots\overset{1}{c}_{11}\overset{1}{d}_{11}) & (\overset{1}{b}_{21}\cdots\overset{1}{c}_{21}\overset{1}{d}_{21}) & \cdots & (\overset{1}{b}_{u^1_1,1}\cdots\overset{1}{c}_{u^1_1,1}\overset{1}{d}_{u^1_1,1}) \\ (\overset{1}{b}_{11}\cdots\overset{1}{c}_{11}\overset{1}{d}_{12}) & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ (\overset{1}{b}_{11}\cdots\overset{1}{c}_{11}\overset{1}{d}_{1n}) & \cdot & \cdots & \cdot \\ (\overset{1}{b}_{11}\cdots\overset{1}{c}_{12}\overset{1}{d}_{11}) & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ (\overset{1}{b}_{1n}\cdots\overset{1}{c}_{1n}\overset{1}{d}_{1n}) & \cdot & \cdots & (\overset{1}{b}_{u^1_1,n}\cdots\overset{1}{c}_{u^1_1,n}\overset{1}{d}_{u^1_1,n}) \end{Vmatrix},$$

$$P = \begin{Vmatrix} \overset{1}{a}_{1,u^1_1+1} & \overset{1}{a}_{1,u^1_1+2} & \cdots & \overset{1}{a}_{1,n} \\ \overset{1}{a}_{2,u^1_1+1} & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \overset{1}{a}_{u^1_1,u^1_1+1} & \cdot & \cdots & \overset{1}{a}_{u^1_1,n} \end{Vmatrix}.$$

Q is a minor of the display $G=(g_{\kappa\lambda})$; $\kappa=\gamma, \cdots, \tau, \epsilon$; $\lambda=j, \cdots, l$ of the matrix $J=B^1\times\cdots\times C^1\times D^1$. By a lemma proved in another paper* by the author, G is non-singular since $B^1, C^1, \cdots, D^1$ are non-singular. Hence Q is non-singular on its columns, and $P=0$. Similarly,

$$\begin{Vmatrix} \overset{1}{b}_{1,u^1_1+1} & \overset{1}{b}_{1,u^1_1+2} & \cdots & \overset{1}{b}_{1,n} \\ \overset{1}{b}_{2,u^1_1+1} & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \overset{1}{b}_{u^1_1,u^1_1+1} & \cdot & \cdots & \overset{1}{b}_{u^1_1,n} \end{Vmatrix} = \cdots = \begin{Vmatrix} \overset{1}{d}_{1,u^1_1+1} & \overset{1}{d}_{1,u^1_1+2} & \cdots & \overset{1}{d}_{1,n} \\ \overset{1}{d}_{2,u^1_1+1} & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \overset{1}{d}_{u^1_1,u^1_1+1} & \cdot & \cdots & \overset{1}{d}_{u^1_1,n} \end{Vmatrix} = 0.$$

Also

$$\left(\sum_{\alpha=1}^{u^1_1} \overset{1}{a}_{\alpha\beta}\overset{1}{b}_{\alpha\gamma}\cdots\overset{1}{d}_{\alpha\epsilon}\right) = \delta,$$

where δ is a δ-matrix on $(\beta, \gamma, \cdots, \tau, \epsilon)$ and $\beta, \gamma, \cdots, \epsilon = 1, \cdots, u^1_1$. Applying Theorem 1 we obtain

$$A^1 = \begin{Vmatrix} \overset{1}{a}_{11} & & 0 & \\ & \ddots & & \\ 0 & & \overset{1}{a}_{u^1_1,u^1_1} & \\ A' & & & A'' \end{Vmatrix}, \quad B^1 = \begin{Vmatrix} \overset{2}{b}_{11} & & 0 & \\ & \ddots & & \\ 0 & & \overset{1}{b}_{u^1_1,u^1_1} & \\ B' & & & B'' \end{Vmatrix}, \cdots,$$

---

* *Composition and rank of n-way matrices and multilinear forms*, Annals of Mathematics, vol. 35 (1934), p. 625.

$$D^1 = \begin{pmatrix} \overset{1}{d}_{11} & & & \\ & \cdot & & 0 \\ & & \cdot & \\ 0 & & \cdot\ \overset{1}{d}_{u_1^1, u_1^1} & \\ D' & & & D'' \end{pmatrix},$$

or matrices obtained from A^1, $B^1, \cdots, D^1$ above by simultaneous interchanges of the first u_1^1 rows and u_1^1 columns of $A^1, \cdots, D^1$.

Assume that $E_1, \cdots, E_{\alpha-1}$ have been reduced to $C_1, \cdots, C_{\alpha-1}$, and the matrices

$$A^{\alpha-1} = (\overset{\alpha-1}{a}_{i\beta}),\ B^{\alpha-1} = (\overset{\alpha-1}{b}_{j\gamma}), \cdots, D^{\alpha-1} = (\overset{\alpha-1}{d}_{l\epsilon}),$$

which satisfy the relations

$$(\overset{\alpha-1}{c}_{ij\cdots l}\overset{\alpha-1}{a}_{i\beta}\overset{\alpha-1}{b}_{j\gamma} \cdots \overset{\alpha-1}{d}_{l\epsilon}) = (\overset{\alpha-1}{c}_{\beta\gamma\cdots\epsilon}) = (\overset{\alpha-1}{c}_{ij\cdots l}),$$

are of the form

$$A^{\alpha-1} = \begin{pmatrix} A^{\alpha-1}_{11} & & & & & & \\ & A^{\alpha-1}_{12} & & & & & \\ & & A^{\alpha-1}_{21} & & & 0 & \\ & 0 & & A^{\alpha-1}_{22} & & & \\ & & & & \ddots & & \\ & & & & & A^{\alpha-1}_{\sigma(\alpha-1)+1,1} & \\ & & A^{\alpha-1}_{\sigma(\alpha-1)+1,3} & & & & A^{\alpha-1}_{\sigma(\alpha-1)+1,2} \end{pmatrix}, \cdots,$$

$$D^{\alpha-1} = \begin{pmatrix} D^{\alpha-1}_{11} & & & & & & \\ & D^{\alpha-1}_{12} & & & & & \\ & & D^{\alpha-1}_{21} & & & 0 & \\ & 0 & & D^{\alpha-1}_{22} & & & \\ & & & & \ddots & & \\ & & & & & D^{\alpha-1}_{\sigma(\alpha-1)+1,1} & \\ & & D^{\alpha-1}_{\sigma(\alpha-1)+1,3} & & & & D^{\alpha-1}_{\sigma(\alpha-1)+1,2} \end{pmatrix},$$

where these matrices satisfy the properties mentioned for A^q in Theorem 15. It is evident at once that E_α can be reduced under transformations with $A^{\alpha-1}, \cdots, D^{\alpha-1}$ to a matrix of type C_α.

We shall now restrict the matrices $A^{\alpha-1}, \cdots, D^{\alpha-1}$ so that

$$(\overset{\alpha}{c}_{ij\cdots l}\overset{\alpha-1}{a}_{i\beta}\overset{\alpha-1}{b}_{j\gamma} \cdots \overset{\alpha-1}{d}_{l\epsilon}) = C_\alpha. \tag{19}$$

Let the orders of the minors $\lambda_1^\alpha, \lambda_2^\alpha, \cdots, \lambda_{\sigma(\alpha)}^\alpha, \delta_\alpha$ be denoted by $u_1^\alpha, u_2^\alpha, \cdots, u_{\sigma(\alpha)}^\alpha, u_{\sigma(\alpha)+1}^\alpha$ respectively; and the orders of the zero minors between these minors by $v_1^\alpha, \cdots, v_{\sigma(\alpha)+1}^\alpha$ respectively. Let $\Sigma_\alpha = u_1^\alpha + \cdots + u_{\sigma(\alpha)}^\alpha + v_1^\alpha + \cdots + v_{\sigma(\alpha)}^\alpha$. By (19),

$$\left(\sum_{\chi=\Sigma_\alpha+1}^{\Sigma_\alpha+u^\alpha_{\sigma(\alpha)+1}} \overset{\alpha-1}{a}_{\chi\beta} \overset{\alpha-1}{b}_{\chi\gamma} \cdots \overset{\alpha-1}{d}_{\chi\epsilon} \right) = 0 \tag{20}$$

if $\gamma, \cdots, \epsilon = \Sigma_\alpha + 1, \cdots, n$, and $\beta = 1, \cdots, \Sigma_\alpha$. By (20) $\Gamma_1\Gamma_2 = 0$, where

$$\Gamma_1 = \begin{Vmatrix} \overset{\alpha-1}{a}_{\Sigma_\alpha+1,1} & \overset{\alpha-1}{a}_{\Sigma_\alpha+2,1} & \cdots & \overset{\alpha-1}{a}_{\Sigma+u^\alpha_{\sigma(\alpha)}+1,1} \\ \overset{\alpha-1}{a}_{\Sigma_\alpha+1,2} & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \overset{\alpha-1}{a}_{\Sigma_\alpha+1,\Sigma_\alpha} & \cdot & \cdots & \overset{\alpha-1}{a}_{\Sigma_\alpha+u^\alpha_{\sigma(\alpha)}+1,\Sigma_\alpha} \end{Vmatrix},$$

$$\Gamma_2 = \begin{Vmatrix} (\overset{\alpha-1}{b}_{\Sigma_\alpha+1,\Sigma_\alpha+1} & \cdots & \overset{\alpha-1}{d}_{\Sigma_\alpha+1,\Sigma_\alpha+1}) & \cdots & (\overset{\alpha-1}{b}_{\Sigma_\alpha+1,n} & \cdots & \overset{\alpha-1}{d}_{\Sigma_\alpha+1,n}) \\ \cdot & & & \cdots & & \cdots & \\ (\overset{\alpha-1}{b}_{\Sigma_\alpha+u^\alpha_{\sigma(\alpha)}+1,\Sigma_\alpha+1} & \cdots & \overset{\alpha-1}{d}_{\Sigma_\alpha+u^\alpha_{\sigma(\alpha)}+1,\Sigma_\alpha+1}) & \cdots & (\overset{\alpha-1}{b}_{\Sigma_\alpha+u^\alpha_{\sigma(\alpha)}+1,n} & \cdots & \overset{\alpha-1}{d}_{\Sigma_\alpha+u^\alpha_{\sigma(\alpha)}+1,n}) \end{Vmatrix}$$

Since $B^{\alpha-1}, \cdots, D^{\alpha-1}$ are to be taken non-singular, the minors

$$B^{\alpha-1}_{\sigma(\alpha-1)+1,2} = \begin{vmatrix} \overset{\alpha-1}{b}_{\Sigma_\alpha+1,\Sigma_\alpha+1} & \overset{\alpha-1}{b}_{\Sigma_\alpha+1,\Sigma_\alpha+2} & \cdots & \overset{\alpha-1}{b}_{\Sigma_\alpha+1,n} \\ \overset{\alpha-1}{b}_{\Sigma_\alpha+2,\Sigma_\alpha+1} & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \overset{\alpha-1}{b}_{n,\Sigma_\alpha+1} & \cdot & \cdots & \overset{\alpha-1}{b}_{n,n} \end{vmatrix}, \cdots,$$

$$D^{\alpha-1}_{\sigma(\alpha-1)+1,2} = \begin{vmatrix} \overset{\alpha-1}{d}_{\Sigma_\alpha+1,\Sigma_\alpha+1} & \overset{\alpha-1}{d}_{\Sigma_\alpha+1,\Sigma_\alpha+1} & \cdots & \overset{\alpha-1}{d}_{\Sigma_\alpha+1,n} \\ \overset{\alpha-1}{d}_{\Sigma_\alpha+2,\Sigma_\alpha+1} & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \overset{\alpha-1}{d}_{n,\Sigma_\alpha+1} & \cdot & \cdots & \overset{\alpha-1}{d}_{n,n} \end{vmatrix}$$

are non-singular. By the lemma of another paper mentioned above,* the display $\Omega = (\omega_{\lambda\mu})$ is non-singular, where $\Omega = B^{\alpha-1}_{\sigma(\alpha-1)+1,2} \times \cdots \times D^{\alpha-1}_{\sigma(\alpha-1)+1,2}$ and $\lambda = j \cdots kl$; $\mu = \gamma \cdots \tau\epsilon$; $\gamma, \cdots, \tau, \epsilon, j, \cdots, k, l = \Sigma_\alpha + 1, \cdots, n$. The

* *Composition and rank of n-way matrices and multilinear forms*, Annals of Mathematics, vol. 35 (1934), p. 625.

matrix Γ_2 is a minor of Ω consisting of certain rows of Ω and is therefore non-singular on its rows. It follows that $\Gamma_1=0$. Similarly

$$\begin{pmatrix} \overset{\alpha-1}{b}_{\Sigma_\alpha+1,1} & \cdots & \overset{\alpha-1}{b}_{\Sigma_\alpha+u^\alpha_{\sigma(\alpha)}+1,1} \\ \cdot & \cdots & \cdot \\ \overset{\alpha-1}{b}_{\Sigma_\alpha+1,\Sigma_\alpha} & \cdots & \overset{\alpha-1}{b}_{\Sigma_\alpha+u^\alpha_{\sigma(\alpha)}+1,\Sigma_\alpha} \end{pmatrix}, \cdots, \begin{pmatrix} \overset{\alpha-1}{d}_{\Sigma_\alpha+1,1} & \cdots & \overset{\alpha-1}{d}_{\Sigma_\alpha+u^\alpha_{\sigma(\alpha)}+1,1} \\ \cdot & \cdots & \cdot \\ \overset{\alpha-1}{d}_{\Sigma_\alpha+1,\Sigma_\alpha} & \cdots & \overset{\alpha-1}{d}_{\Sigma_\alpha+u^\alpha_{\sigma(\alpha)}+1,\Sigma_\alpha} \end{pmatrix}$$

are zero

It follows from (19) that

$$\left(\sum_{\lambda=\Sigma_\alpha+1}^{\Sigma_\alpha+u_{\sigma(\alpha)}+1} \overset{\alpha-1}{a}_{\lambda\beta} \cdots \overset{\alpha-1}{d}_{\lambda\epsilon}\right) = 0 \tag{21}$$

if $\beta=\Sigma_\alpha+u^\alpha_{\sigma(\alpha)+1}+1,\ \Sigma_\alpha+u^\alpha_{\sigma(\alpha)+1}+2, \cdots, n;\ \gamma, \cdots, \tau,\ \epsilon=\Sigma_\alpha+1, \cdots, n$.

By (21), $\Gamma_3\Gamma_2=0$, where

$$\Gamma_3 = \begin{pmatrix} \overset{\alpha-1}{a}_{\Sigma_\alpha+1,\Sigma_\alpha+u^\alpha_{\sigma(\alpha)}+1+1} & \cdots & \overset{\alpha-1}{a}_{\Sigma_\alpha+u^\alpha_{\sigma(\alpha)}+1,\Sigma_\alpha+u^\alpha_{\sigma(\alpha)}+1+1} \\ \cdot & \cdots & \cdot \\ \overset{\alpha-1}{a}_{\Sigma_\alpha+1,n} & \cdots & \overset{\alpha-1}{a}_{\Sigma_\alpha+u^\alpha_{\sigma(\alpha)}+1,n} \end{pmatrix}.$$

Again, since Γ_2 is non-singular on its rows, $\Gamma_3=0$. Similarly,

$$\begin{pmatrix} \overset{\alpha-1}{b}_{\Sigma_\alpha+1,\Sigma_\alpha+u^\alpha_{\sigma(\alpha)}+1+1} & \cdots & \overset{\alpha-1}{b}_{\Sigma_\alpha+u^\alpha_{\sigma(\alpha)}+1,\Sigma_\alpha+u^\alpha_{\sigma(\alpha)}\ 1+1} \\ \cdot & \cdots & \cdot \\ \overset{\alpha-1}{b}_{\Sigma_\alpha+1,n} & \cdots & \overset{\alpha-1}{b}_{\Sigma_\alpha+u^\alpha_{\sigma(\alpha)}+1,n} \end{pmatrix} = \cdots$$

$$= \begin{pmatrix} \overset{\alpha-1}{d}_{\Sigma_\alpha+1,\Sigma_\alpha+u^\alpha_{\sigma(\alpha)}+1+1} & \cdots & \overset{\alpha-1}{d}_{\Sigma_\alpha+u^\alpha_{\sigma(\alpha)}+1,\Sigma_\alpha+u^\alpha_{\sigma(\alpha)}+1+1} \\ \cdot & \cdots & \cdot \\ \overset{\alpha-1}{d}_{\Sigma_\alpha+1,n} & \cdots & \overset{\alpha-1}{d}_{\Sigma_\alpha+u^\alpha_{\sigma(\alpha)}+1,n} \end{pmatrix} = 0.$$

Further if $\beta, \gamma, \cdots, \epsilon=\Sigma_\alpha+1, \cdots, \Sigma_\alpha+u^\alpha_{\sigma(\alpha)+1}$,

$$\left(\sum_{\lambda=\Sigma_\alpha+1}^{\Sigma_\alpha+u_{\sigma(\alpha)}+1+1} \overset{\alpha-1}{a}_{\lambda\beta} \cdots \overset{\alpha-1}{d}_{\lambda\epsilon}\right) = \delta, \tag{22}$$

where δ is a δ-matrix on $(\beta, \gamma, \cdots, \epsilon)$. Equations of type (22) have already been treated.*

In view of the above considerations, the matrices $A^{\alpha-1}, \cdots, D^{\alpha-1}$ which satisfy (19) are of the form $A^\alpha, \cdots, D^\alpha$ as obtained from (17) by writing $q=\alpha$, and satisfy the properties of Theorem 15 for these matrices.

Theorems 14 and 15 follow by induction.

* See Theorem 1.

We shall now prove an analogue of Theorems 12 and 13. Let $C_{\alpha 1}=(c'_{\beta\gamma\cdots\epsilon})$ be a matrix obtained from C_α of Theorem 14 by replacing δ_α by a zero matrix. We have

THEOREM 16. *Let $\Sigma=(C_1, \cdots, C_{\alpha-1}, K_\alpha)$ be a set of p-way, $p\geqq 3$, matrices of order n with elements in a field ϕ, where $C_1, \cdots, C_{\alpha-1}$ are canonical diagonal matrices as given in Theorem 14. If the set Σ is equivalent in ϕ to a set of diagonal matrices, the orders of $\lambda_1^\alpha, \lambda_2^\alpha, \cdots, \lambda^\alpha_{\sigma(\alpha)}, \delta_\alpha$ and values of the non-vanishing elements of $C_{\alpha 1}$ can be chosen so that $K_\alpha - C_{\alpha 1}$ is a non-singular matrix of order r, where r is the order of δ_α, or such a matrix bordered by zeros; and conversely.*

Let $K_\alpha=(k_{\beta\gamma\cdots\epsilon})$. If Σ is equivalent to a set of diagonal matrices, there exist matrices $A^{\alpha-1}=(a^{\alpha-1}_{i\beta}), \cdots, D^{\alpha-1}=(d^{\alpha-1}_{l\epsilon})$, leaving $C_1, \cdots, C_{\alpha-1}$ invariant, and minors $\lambda_1^\alpha, \cdots, \lambda^\alpha_{\sigma(\alpha)}, \delta_\alpha$ such that

$$(k_{\beta\gamma\cdots\epsilon}) = (\overset{\alpha}{c}_{ij\cdots l}\overset{\alpha-1}{a_{i\beta}}\overset{\alpha-1}{b_{j\gamma}}\cdots \overset{\alpha-1}{d_{l\epsilon}}),$$

whence

$$\begin{aligned}(k_{\beta\gamma\cdots\epsilon}) = \Bigg(&\sum_{\lambda=1}^{\overset{\alpha}{u}_1} \overset{\alpha}{c}_{\lambda\cdots\lambda}\overset{\alpha-1}{a_{\lambda\beta}}\cdots \overset{\alpha-1}{d_{\lambda\epsilon}} + \sum_{\lambda=\overset{\alpha}{u}_1+\overset{\alpha}{v}_1+1}^{\overset{\alpha}{u}_1+\overset{\alpha}{v}_1+\overset{\alpha}{u}_2} \overset{\alpha}{c}_{\lambda\cdots\lambda}\overset{\alpha-1}{a_{\lambda\beta}}\cdots \overset{\alpha-1}{d_{\lambda\epsilon}} \\ &+\cdots+ \sum_{\lambda=\Sigma_\alpha-\overset{\alpha}{u}_{\sigma(\alpha)}-\overset{\alpha}{v}_{\sigma(\alpha)}+1}^{\Sigma_\alpha-\overset{\alpha}{v}_{\sigma(\alpha)}} \overset{\alpha}{c}_{\lambda\cdots\lambda}\overset{\alpha-1}{a_{\lambda\beta}}\cdots \overset{\alpha-1}{d_{\lambda\epsilon}} \\ &+ \sum_{\lambda=\Sigma_\alpha+1}^{\Sigma_\alpha+\overset{\alpha}{u}_{\sigma(\alpha)}+1} \overset{\alpha-1}{a_{\lambda\beta}}\cdots \overset{\alpha-1}{d_{\lambda\epsilon}}\Bigg),\end{aligned}$$

where the u's and v's are as defined in the proof of Theorems 14 and 15. At once

$$K_\alpha - C_{\alpha 1} = \Bigg(\sum_{\lambda=\Sigma_\alpha+1}^{\Sigma_\alpha+\overset{\alpha}{u}_{\sigma(\alpha)}+1} \overset{\alpha-1}{a_{\lambda\beta}}\cdots \overset{\alpha-1}{d_{\lambda\epsilon}}\Bigg).$$

If we wish to test the equivalence of a set $C_1, \cdots, C_{\alpha-1}, K_\alpha, \cdots, K_m$ to diagonal matrices, we determine whether or not $C_1, \cdots, C_{\alpha-1}, K_\alpha$ is equivalent to a set $C_1, \cdots, C_\alpha$. If so, let $K_{\alpha+1}, \cdots, K_m$ go into $K'_{\alpha+1}, \cdots, K'_m$ under transformation of the set $C_1, \cdots, C_{\alpha-1}, K_\alpha$ to $C_1, \cdots, C_\alpha$. The matrices $C_1, \cdots, C_{\alpha-1}, K_\alpha, \cdots, K_m$ are equivalent under non-singular linear transformations to diagonal matrices if and only if $C_1, \cdots, C_\alpha, K'_{\alpha+1}, \cdots, K'_m$ are equivalent to diagonal matrices. We reapply Theorem 16 to $C_1, \cdots, C_\alpha, K'_{\alpha+1}$. This process can be continued until we arrive at a canonical set $C_1, \cdots, C_m$.

In Theorem 14 we obtained canonical forms of p-way diagonal matrices, $p\geqq 3$, and in Theorem 16 necessary and sufficient conditions for the equivalence of $C_1, \cdots, C_{\alpha-1}, K_\alpha$ to canonical diagonal matrices. Since a set of matrices is equivalent to a set of diagonal matrices if and only if it is equivalent to a set of canonical diagonal matrices, we have derived necessary and sufficient conditions for the equivalence of a set of p-way matrices, $p \geqq 3$, to a set of diagonal matrices.

Theorem 16 is in general difficult to apply. However, it is given here since no better equivalent theorem has been found.

Chapter III. Factorization of p-way matrices into 3-way matrices

11. **Introduction.** Necessary and sufficient conditions for a matrix $A=(a_{k_1\cdots k_p})$ to be of the form $(a^{(1)}_{\alpha k_1}\cdots a^{(p)}_{\alpha k_p})$, where $(a^{(1)}_{\alpha k_1}), \cdots, (a^{(p)}_{\alpha k_p})$ are matrices non-singular on the index α, are given in chapter I. In the present chapter necessary and sufficient conditions are obtained for a matrix $A=(a_{ijk_1\cdots k_p})$ to be of the form $(a^{(1)}_{ijk_1}\cdots a^{(p)}_{ijk_p})$, i not summed, j summed, where the 3-way matrices $(a^{(1)}_{ijk_1}), \cdots, (a^{(p)}_{ijk_p})$ are non-singular on ij. The method of treatment covers the case where the index j in the matrices above does not occur* and the matrices $(a^{(1)}_{ik_1}), \cdots, (a^{(p)}_{ik_p})$ are non-singular on i.

In this chapter, as in the others, indices may be partitions consisting of more than one index.

12. **Factorization into multiple composites.** Let $\delta_1, \cdots, \delta_{m-1}$ designate the 2-way displays $(\delta^1_{TT'})=(\delta_{1i_1i_2}\delta_{j_1j_2})$, $(\delta^2_{TT'})=(\delta_{2i_1i_2}\delta_{j_1j_2}), \cdots, (\delta^{m-1}_{TT'})$ $=(\delta_{m-1,i_1i_2}\delta_{j_1j_2})$ respectively, where $T=i_1j_1$, $T'=i_2j_2$; $(\delta_{j_1j_2})$ is a Kronecker delta of order t, and $(\delta_{1i_1i_2}), \cdots, (\delta_{m-1,i_1i_2})$ are the i-layers of a δ-matrix on (i, i_1, i_2) of order m obtained by setting $i=1, \cdots, m-1$ respectively. Let $\delta_2' =(\delta'_{1i_1i_2}\delta_{j_1j_2})$, $\delta_3' =(\delta'_{2i_1i_2}\delta_{j_1j_2}), \cdots, \delta'_{m-1}=(\delta'_{m-2,i_1i_2}\delta_{j_1j_2})$, where $(\delta'_{1i_1i_2})$, $(\delta'_{2i_1i_2}), \cdots, (\delta'_{m-2,i_1i_2})$ are the i-layers of the δ-matrix $(\delta'_{ii_1i_2})$ on (i, i_1, i_2) of order $(m-1)$ on each index obtained by setting $i=1, \cdots, m-2$ respectively.

Let $B_1, \cdots, B_{m-1}$ be a set of 2-way matrices of order $n=mt$ with elements in a field ϕ. If the matrices $B_1, \cdots, B_{m-1}$ are equivalent under similar transformation in ϕ to $\delta_1, \cdots, \delta_{m-1}$, it is evidently necessary that the matrices B_i be each equivalent under similar transformation in ϕ to δ_i for $i=1, \cdots, m-1$. Now B_1 is equivalent under similar transformation to δ_1 if and only if $(B_1-\lambda I)$ has the same invariant factors† as $(\delta_1-\lambda I)$, where I is a Kronecker delta. Assume that B_1 is equivalent to δ_1 as demanded. By reduction of B_1 to δ_1

* The matrix A in this case is a generalization of the Scott product of two matrices. See M. Lecat, *Abrégé de la Théorie des Déterminants à n Dimensions*, 1911, Introduction, p. xl.

† Dickson, *Modern Algebraic Theories*, p. 104.

under similar transformation, the matrices $B_2, \cdots, B_{m-1}$ are transformed into a set $B_2', \cdots, B_{m-1}'$. Let

$$B_s' = \begin{pmatrix} B_{s11}' & B_{s12}' \\ B_{s21}' & B_{s22}' \end{pmatrix} \qquad (s = 2, 3, \cdots, m-1),$$

where B_{s11} is a minor of order t.

THEOREM 17. *The matrices $B_1, \cdots, B_{m-1}$ of order n with elements in a field ϕ are equivalent under similar transformation in ϕ to a set $\delta_1, \cdots, \delta_{m-1}$ if and only if*

$$B_{s11}' = B_{s21}' = B_{s12}' = 0 \qquad (s = 2, \cdots, m-1),$$

and the set $B_{222}', B_{322}', \cdots, B_{m-1,22}'$ is equivalent under similar transformation in ϕ to the set of $(n-t)$-order matrices $\delta_2', \delta_3', \cdots, \delta_{m-1}'$.

The matrix W which satisfies the relation $W\delta_1 W^{-1} = \delta_1$ is of the form*

$$W = \begin{pmatrix} W_{11} & 0 \\ 0 & W_{22} \end{pmatrix},$$

where W_{11} is a square minor of W of order t. Now for $s = 2, \cdots, m-1$,

$$WB_s' W^{-1} = \begin{pmatrix} W_{11}B_{s11}'W_{11}^{-1} & W_{11}B_{s12}'W_{22}^{-1} \\ W_{22}B_{s21}'W_{11}^{-1} & W_{22}B_{s22}'W_{22}^{-1} \end{pmatrix}.$$

Equating $WB_s' W^{-1}$ above to δ_s we obtain the conditions of Theorem 17. The matrices $\delta_2', \delta_3', \cdots, \delta_{m-1}'$ form an array like $\delta_1, \delta_2, \cdots, \delta_{m-1}$, whence the above process may be reapplied to $B_{222}', B_{322}', \cdots, B_{m-1,22}'$, and $\delta_2', \delta_3', \cdots, \delta_{m-1}'$. Since m is finite this process is a terminating one.

Let $\xi_i, i = 1, \cdots, n-1$, now represent a diagonal matrix with the ith element on the diagonal as the only non-vanishing element. For square 2-way matrices $B_1, \cdots, B_{n-1}$ of order n to be equivalent under similar transformation to the set $\xi_1, \cdots, \xi_{n-1}$ it is in particular necessary that $(B_1 - \lambda I)$ have the invariant factors $\lambda(\lambda-1), \lambda, \cdots, \lambda$. Assuming that this condition is satisfied, let the set $B_1, \cdots, B_{n-1}$ be reduced under similar transformation in ϕ to a set $\xi_1, B_2', \cdots, B_{n-1}'$, where we write

$$B_i' = \begin{pmatrix} b_{i11}' & B_{i12}' \\ B_{i21}' & B_{i22}' \end{pmatrix} \qquad (i = 2, \cdots, n-1),$$

where b_{i11}' is a single element. Letting $n = mt, t = 1$ in Theorem 17, we have the

* Turnbull and Aitken, *Introduction to the Theory of Canonical Matrices*, 1932, p. 146.

COROLLARY. *The 2-way matrices* $B_1, \cdots, B_{n-1}$ *of order* n *are equivalent under similar transformation in* ϕ *to* $\xi_1, \cdots, \xi_{n-1}$, *if and only if* $b'_{i11}=B'_{i12}=B'_{i21}=0$, $i=2, 3, \cdots, n-1$, *and the set* $B'_{222}, B'_{322}, \cdots, B'_{n-1,22}$ *is equivalent under similar transformation in* ϕ *to the set* $\xi'_2, \xi'_3, \cdots, \xi'_{n-1}$ *where* ξ'_i, $i=2, \cdots, n-1$, *are* $(n-1)$ *by* $(n-1)$ *diagonal matrices which possess the* $(i-1)$*st diagonal element as the only non-vanishing element, it being unity.*

Let $\delta_1, \delta_2, \cdots, \delta_{m-1}$ now denote the matrices $(\delta_{1i_1\cdots i_p})\times(\delta_{j_1\cdots j_p})$, $(\delta_{2i_1\cdots i_p})\times(\delta_{j_1\cdots j_p}), \cdots, (\delta_{m-1,i_1\cdots i_p})\times(\delta_{j_1\cdots j_p})$ respectively, where $(\delta_{j_1\cdots j_p})$ is a δ-matrix on $(j_1, \cdots, j_p)$ of order t, and $(\delta_{1,i_1\cdots i_p}), \cdots, (\delta_{m-1,i_2\cdots i_p})$ are i-layers of a δ-matrix on $(i, i_1 \cdots i_p)$ of order m obtained by setting $i=1, \cdots, m-1$ respectively. Let $B_1, \cdots, B_{m-1}$ be p-way matrices, $p \geqq 3$, of order n, where $n=mt$, and where the matrix $B_1=(b_{k_1\cdots k_p})$ is non-singular. Let the matrices $C_s=(c^s_{k_si_sj_s})$; $s=1, \cdots, p$; $k_s=1, \cdots, n$, $i_s=1, \cdots, m$; $j_s=1, \cdots, t$, non-singular on k_s, i_sj_s (i_sj_s is a single partition), reduce B_1 to δ under similar transformation, where δ is a δ-matrix on $(i_1j_1, i_2j_2, \cdots, i_pj_p)$; let these matrices simultaneously reduce $B_2, \cdots, B_{m-1}$ to $B'_2, \cdots, B'_{m-1}$. With these notations we can state

THEOREM 18. *The set of p-way matrices* $B_1, \cdots, B_{m-1}$, $p \geqq 3$, *of order* n *is equivalent in* ϕ *under similar transformation on* $k_1, \cdots, k_p$ *with* $C_1=(c^{(1)}_{k_1i_1j_1})$, $\cdots, C_p=(c^{(p)}_{k_pi_pj_p})$ *to a set* $\delta_1, \cdots, \delta_{m-1}$ *if and only if* $B'_s=\delta_s$ *for* $s=2, \cdots, m-1$ *or* $B''_s=\delta_s$ *where* $B''_2, \cdots, B''_{m-1}$ *are obtained from* $B'_2, \cdots, B'_{m-1}$ *by simultaneously rearranging the* $(i_1j_1, i_2j_2, \cdots, i_pj_p)$ *diagonal elements of* $B'_2, \cdots, B'_{m-1}$.

It has been shown in Theorem 2 that under similar transformation on the partitions $i_1j_1, i_2j_2, \cdots, i_pj_p$ the $(i_1j_1, i_2j_2, \cdots, i_pj_p)$ diagonal elements of a matrix are at most rearranged, whence Theorem 18 follows.

Let $\xi_1, \cdots, \xi_n$ now denote the i-layers of a δ-matrix on $(i, k_1, \cdots, k_p)$ of order n. Let matrices $B_s=(b_{i_1i_2\cdots i_p})$, $s=1, \cdots, n-1$, be of order n with elements in a field ϕ. We have the

COROLLARY. *The set of p-way matrices* $B_1, \cdots, B_{n-1}$, $p \geqq 3$, *of order* n *is equivalent under similar transformation in* ϕ *on* $i_1, \cdots, i_p$ *with matrices* $(c^{(1)}_{i_1k_1}), \cdots, (c^{(p)}_{i_pk_p})$ *to* $\xi_1, \cdots, \xi_{n-1}$ *if and only if* $B_i=\xi_i$ *for* $i=1, \cdots, n-1$ *or* $B''_i=\xi_i$, $i=1, \cdots, n-1$, *where* $B''_i, \cdots, B''_{n-1}$ *are obtained by simultaneously reordering the* $(i_1, \cdots, i_p)$ *diagonal elements of* $B_1, \cdots, B_{n-1}$.

We now prove

THEOREM 19. *Given a matrix $B=(b_{ik_1\cdots k_p})$ with elements in a field ϕ where $i=1,\cdots,m$ and $k_1,\cdots,k_p=1,\cdots,n=mt$. Let $B_1,\cdots,B_m$ be the i-layers of B obtained by letting $i=1,\cdots,m$ respectively. Let δ be a δ-matrix on $(j_1,\cdots,j_p)$ of order t, and $\delta_1'',\cdots,\delta_m''$ be the i-layers of a δ-matrix δ'' on $(i,i_1,\cdots,i_p)$ of order m obtained by letting $i=1,\cdots,m$ respectively. The matrix B is the multiple composite on i, j of matrices $A_s=(a^s_{ijk_s})$, $s=1,\cdots,p$, non-singular on ij, k_s, with elements in ϕ if and only if there exist matrices $C_s=(c^s_{k_si_sj_s})$ non-singular on k_s, i_sj_s with elements in ϕ such that*

$$B_1 \mid k_1k_2\cdots k_p \mid C_1\times C_2\times\cdots\times C_p = \delta_1''\times\delta,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$B_m \mid k_1k_2\cdots k_p \mid C_1\times C_2\times\cdots\times C_p = \delta_m''\times\delta.$$

If $C=(a^{(1)}_{ijk_1}\cdots a^{(p)}_{ijk_p})$, i not summed, where $A_s=(a^s_{i_sj_sk_s})$; $i=i_s=1,\cdots,m$; $j=j_s=1,\cdots,t$; $k_1,\cdots,k_p=1,\cdots,n=mt$, then

$$C = \delta''\times\delta \mid i_1\cdots i_pj_1\cdots j_p \mid A_1\times\cdots\times A_p. \tag{23}$$

If the matrix $B=(b_{ik_1\cdots k_p})=C$, then by (23)

$$\begin{aligned} &B \mid k_1\cdots k_p \mid A_1^{-1}\times\cdots\times A_p^{-1} \\ &\quad = \delta''\times\delta \mid i_1\cdots i_pj_1\cdots j_p \mid (\delta_{i_1j_1i_1'j_1'})\times\cdots\times(\delta_{i_pj_pi_p'j_p'}), \end{aligned} \tag{24}$$

where $A_s^{-1}=(A_{k_si_s'j_s'})$ is the inverse of the matrix A_s on i_sj_s, k_s and $(\delta_{i_sj_si_s'j_s'})$, displayed as $(\delta_{T_sT_s'})$, $T_s=i_sj_s$, $T_s'=i_s'j_s'$, is a Kronecker delta.

The author has shown* that

$$(\delta_{i_sj_si_s'j_s'}) = (\delta_{i_si_s'})\times(\delta_{j_sj_s'}) \qquad (s=1,\cdots,p), \tag{25}$$

where $(\delta_{i_si_s'})$, $(\delta_{j_sj_s'})$ are Kronecker deltas. By (24) and (25) we have

$$B \mid k_1\cdots k_p \mid A_1^{-1}\times\cdots\times A_p^{-1} = \delta''\times\delta. \tag{26}$$

Equation (26) is equivalent to the set of equations

$$\begin{aligned} &B_1 \mid k_1\cdots k_p \mid A_1^{-1}\times\cdots\times A_p^{-1} = \delta_1''\times\delta, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &B_m \mid k_1\cdots k_p \mid A_1^{-1}\times\cdots\times A_p^{-1} = \delta_m''\times\delta, \end{aligned} \tag{27}$$

where $\delta_1'',\cdots,\delta_m''$ are the i-layers of δ''. This proves the theorem.

To show that the factorization property of Theorem 19 can be recognized by Theorems 17 and 18, we note that if the i-layers $B_1,\cdots,B_m$ of B are equivalent under non-singular linear transformations to $\delta=\delta_1''\times\delta,\cdots,\delta_m=\delta_m''\times\delta$

* *Composition and rank of n-way matrices and multilinear forms*, Annals of Mathematics, vol. 35 (1934), p. 629.

as in (27), then $\bar{B}=B_1+\cdots+B_m$, $B_1,\cdots,B_{m-1}$ are equivalent under non-singular linear transformations to $\delta=(\delta_1''\times\delta+\cdots+\delta_m''\times\delta)$, $\delta_1,\cdots,\delta_{m-1}$, where $\delta_1,\cdots,\delta_{m-1}$ are defined as in Theorem 18, and conversely. Now δ is a δ-matrix on the partitions $(i_1j_1,\cdots,i_pj_p)$. If $\bar{B}$ is non-singular, reduce $\bar{B}$ to δ under non-singular linear transformations. Simultaneously $B_1,\cdots,B_{m-1}$ transform into matrices $B_1',\cdots,B_{m-1}'$. The matrices $\bar{B}$, $B_1,\cdots,B_{m-1}$ are equivalent to δ, $\delta_1,\cdots,\delta_{m-1}$ if and only if $B_1',\cdots,B_{m-1}'$ are equivalent under similar transformation to $\delta_1,\cdots,\delta_{m-1}$. The conditions for such equivalence for the 2-way case are given in Theorem 17 and for the p-way case, $p\geqq 3$, in Theorem 18.

COROLLARY. *Let $\xi_1,\cdots,\xi_n$ be the i-layers of a δ-matrix on $(i,i_1,\cdots,i_p)$ of order n, where these layers are obtained by letting $i=1,\cdots,n$ respectively. The matrix $B=(b_{ik_1\cdots k_p})$ of order n with elements in a field ϕ can be written in the form $(a_{ik_1}^{(1)}\cdots a_{ik_p}^{(p)})$, i not summed, where $(a_{ik_1}^{(1)}),\cdots,(a_{ik_p}^{(p)})$ are non-singular and possess elements in ϕ, if and only if the i-layers $B_1=(b_{1k_1\cdots k_p})$, $B_2=(b_{2k_1\cdots k_p}),\cdots,B_n=(b_{nk_1\cdots k_p})$ of B are equivalent in ϕ to $\xi_1,\xi_2,\cdots,\xi_n$.*

Consider the matrix $B=(b_{ik_1\cdots k_p})$, $p\geqq 3$, of order m on i, and order $n=mt$ on $k_1,\cdots,k_p$. Let $\delta_1,\cdots,\delta_{m-1}$ be again defined as in Theorem 18. If $\bar{B}=(\sum_{i=1}^m b_{ik_1\cdots k_p})$ is non-singular, the matrices $E_s=(e^s_{i_sj_sk_s})$, non-singular on k_s,i_sj_s, which reduce $\bar{B}$ to a δ-matrix on $(i_1j_1,\cdots,i_pj_p)$; $i_1,\cdots,i_p=1,\cdots,m$; $j_1,\cdots,j_p=1,\cdots,t$, reduce the i-layers $B_1,\cdots,B_{m-1}$ of B, obtained by setting $i=1,\cdots,m-1$ respectively, to a set $B_1',\cdots,B_{m-1}'$. Theorems 2, 18, and 19 imply

THEOREM 20. *For $B=(b_{ik_1\cdots k_p})$, $p\geqq 3$, of order m on i and $n=mt$ on $k_1,\cdots,k_p$, to be the multiple composite on i, j of the matrices $A_s=(a^s_{ijk_s})$, $s=1,\cdots,p$, non-singular on ij, k_s, it is necessary that the sum $\bar{B}$ of the i-layers of B be non-singular. It is further necessary that*

$$B_1'=\delta_1,\cdots,B_{m-1}'=\delta_{m-1},$$

or

$$B_1''=\delta_1,\cdots,B_{m-1}''=\delta_{m-1},$$

where $B_1'',\cdots,B_{m-1}''$ are obtained from $B_1',\cdots,B_{m-1}'$ by simultaneous rearrangements of the $(i_1j_1,\cdots,i_pj_p)$ diagonal elements of $B_1',\cdots,B_{m-1}'$. These conditions are also sufficient.

Now let B represent the matrix $(b_{ik_1\cdots k_p})$, $p\geqq 3$, of order n. Let the i-layers of B obtained by setting $i=1,\cdots,n$ be denoted by $B_1,\cdots,B_n$ respectively. If $\bar{B}=(\sum_{i=1}^n b_{ik_1\cdots k_p})$ is non-singular, reduce $\bar{B}$, $B_1,\cdots,B_{n-1}$ by means of non-singular linear transformations to $\xi, B_1', \cdots, B_{n-1}'$, where ξ is a δ-matrix on $(i_1, \cdots, i_p)$ of order n. As in the corollary of Theorem 18 let $\xi_1, \cdots, \xi_n$ be the i-layers of a δ-matrix on $(i, i_1, \cdots, i_p)$ of order n. We have the following

COROLLARY. *For $B=(b_{ik_1\cdots k_p})$, $p\geqq 3$, of order n with elements in a field ϕ to be factorable into the form $K=(a_{ik_1}^{(1)}\cdots a_{ik_p}^{(p)})$, i not summed, where $(a_{ik_1}^{(1)}), \cdots (a_{ik_p}^{(p)})$ are non-singular with elements in ϕ, it is necessary that $\overline{B}$ be non-singular. It is further necessary that $B_s'=\xi_s$; $s=1, \cdots, n-1$, or $B_s''=\xi_s$; $s=1, \cdots, n-1$, where $B_1'', \cdots, B_{n-1}''$ are obtained from $B_1', \cdots, B_{n-1}'$ by simultaneous rearrangements of the $(i_1, \cdots, i_p)$ diagonal elements of $B_1', \cdots, B_{n-1}'$. These conditions are also sufficient.*

Theorems 17 to 20 may be extended at once to the multiple composite on i, j of $A_1=(a_{ijk_1}^{(1)}), \cdots, A_p=(a_{ijk_p}^{(p)})$ where $A_1, \cdots, A_p$ are non-singular on ij only.

The matrix $\delta''\times\delta$ above is a canonical matrix of a class of $(p+1)$-way matrices. We have hence determined necessary and sufficient conditions for the equivalence of a $(p+1)$-way matrix to such a diagonal matrix under non-singular linear transformations on all but one index.

APPENDIX

For certain situations, the equivalence under similar transformation in a given field of a set of 2-way matrices $S=(T_1, \cdots, T_m)$ of order n to a set of diagonal matrices can be recognized by the rank of a matrix H associated with S. This appendix is devoted to the derivation of H.

Let

$$C_i = \begin{pmatrix} c_1{}^i & & 0 \\ & \ddots & \\ 0 & & c_n{}^i \end{pmatrix}$$

be a classical canonical matrix† to which T_i is similar for $i=1, \cdots, m$.

Let α_i, $i=1, \cdots, n$, denote the matrix

$$\begin{pmatrix} T_1' - d_i^1 I \\ \vdots \\ T_m' - d_i^m I \end{pmatrix},$$

where $T_1', \cdots, T_m'$ are the transposes of $T_1, \cdots, T_m$ and $d_1^\sigma, \cdots, d_n^\sigma$ are permutations of $c_1{}^\sigma, \cdots, c_n{}^\sigma$ for $\sigma=1, \cdots, m$. I is a Kronecker delta. *Assume that the matrices α_i, $i=1, \cdots, n$, are all of rank $n-1$.* Then there exist

† i in $c_j{}^i$ is a superscript.

values $\psi_{i1}, \psi_{i2}, \cdots, \psi_{i,n(m-1)+1}$ of ψ_i and a value of ξ_i such that the minor $A_{i1}, i=1,\cdots,n$, obtained from α_i by deleting the ψ_{i1}th, ψ_{i2}th, $\cdots, \psi_{i,n(m-1)+1}$th rows and the ξ_ith column is a non-singular minor of order $n-1$. Let A_{2i}, $i=1, \cdots, n$, be the ξ_ith column of α_i with the ψ_{i1}th, ψ_{i2}th, $\cdots, \psi_{i,n(m-1)+1}$th elements deleted. Let $H_{11}, \cdots, H_{1n}, H_{21}, \cdots, H_{2n}$ be defined by the relations

$$\begin{pmatrix} H_{11} \\ H_{21} \end{pmatrix} = - A_{11}^{-1} A_{21}, \cdots, \begin{pmatrix} H_{1n} \\ H_{2n} \end{pmatrix} = - A_{1n}^{-1} A_{2n},$$

where H_{1i} is a column composed of the first $\xi_i - 1$ elements of the column $-A_{1i}^{-1}A_{2i}$, and H_{2i} is a column composed of the remaining elements of $-A_{1i}^{-1}A_{2i}$. Let H now denote the matrix

$$\begin{pmatrix} H_{11} & & H_{1n} \\ 1 & \cdots & 1 \\ H_{21} & & H_{2n} \end{pmatrix},$$

where the unit element in the ith column of H occurs in the ξ_ith row for $i=1, \cdots, n$.

Evidently the set S is equivalent under similar transformation to a set of diagonal matrices if and only if the set S is equivalent under similar transformation to the set $S' = (C_1, \cdots, C_m)$ or $S^* = (C_1^*, \cdots, C_m^*)$, where $C_1^*, \cdots, C_m^*$ are matrices obtained from $C_1, \cdots, C_m$ by arbitrary interchanges of the diagonal elements. If the set S is to be equivalent under similar transformation to the set S', there exists a non-singular matrix $X = (x_{ij})$ such that

$$XT_1X^{-1} = C_1, \cdots, XT_mX^{-1} = C_m,$$

or what is the same thing,

$$XT_1 = C_1X, \cdots, XT_m = C_mX. \tag{1}$$

Equations (1) are equivalent to the set of equations

$$\alpha_1 \begin{pmatrix} x_{11} \\ \cdot \\ \cdot \\ \cdot \\ x_{1,} \end{pmatrix} = 0, \tag{2_1}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\alpha_n \begin{pmatrix} x_{n1} \\ \cdot \\ \cdot \\ \cdot \\ x_{n,} \end{pmatrix} = 0. \tag{2_n}$$

Since α_1 is of rank $n-1$, equation (2_1) can be written as

$$A_{11}\begin{pmatrix} x_{11} \\ \cdot \\ \cdot \\ \cdot \\ x_{1,\xi_1-1} \\ x_{1,\xi_1+1} \\ \cdot \\ \cdot \\ \cdot \\ x_{1n} \end{pmatrix} = -A_{21}x_{1,\xi_1}.$$

Now

$$\begin{pmatrix} x_{11} \\ \cdot \\ \cdot \\ \cdot \\ x_{1,\xi_1-1} \\ x_{1,\xi_1+1} \\ \cdot \\ \cdot \\ \cdot \\ x_{1n} \end{pmatrix} = -A_{11}^{-1}A_{21}x_{1,\xi_1}.$$

Since $\alpha_2, \cdots, \alpha_n$ are all of rank $n-1$, we obtain similar solutions from $(2_2), \cdots, (2_n)$ of the form

$$\begin{pmatrix} x_{21} \\ \cdot \\ \cdot \\ \cdot \\ x_{2,\xi_2-1} \\ x_{2,\xi_2+1} \\ \cdot \\ \cdot \\ \cdot \\ x_{2n} \end{pmatrix} = -A_{12}^{-1}A_{22}x_{2,\xi_2}, \cdots, \begin{pmatrix} x_{n1} \\ \cdot \\ \cdot \\ \cdot \\ x_{n,\xi_n-1} \\ x_{n,\xi_n+1} \\ \cdot \\ \cdot \\ \cdot \\ x_{nn} \end{pmatrix} = -A_{1n}^{-1}A_{2n}x_{n,\xi_n}.$$

The matrix X can now be written as

$$X = \begin{pmatrix} H_{11}x_{1,\xi_1} & & H_{1n}x_{n,\xi_n} \\ x_{1,\xi_1} & \cdots & x_{n,\xi_n} \\ H_{21}x_{1,\xi_1} & & H_{2n}x_{n,\xi_n} \end{pmatrix}.$$

Evidently

$$X = H\begin{pmatrix} x_{1,\xi_1} & & 0 \\ & \ddots & \\ 0 & & x_{n,\xi_n} \end{pmatrix},$$

whence X can be taken non-singular if and only if H is non-singular.

A like argument holds if the set S' is replaced in the equations of (1) by S^*. We have proved the

THEOREM. *The set S is equivalent under similar transformation to a set of diagonal matrices if and only if H is non-singular for at least one choice of the quantities* $d_1^1, \cdots, d_n^1, d_1^2, \cdots, d_n^2, \cdots, d_1^m, \cdots, d_n^m$.

ARMOUR INSTITUTE OF TECHNOLOGY,
CHICAGO, ILL.